GRAPHS WITH THE $k$-LINKED AND RELATED PROPERTIES

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Abstract. A graph is called $k$-linked if it has at least $2k$ vertices and the property that for every sequence $s_1, \ldots, s_k, t_1, \ldots, t_k$, there exist disjoint paths $P_1, \ldots, P_k$ such that $s_i$ and $t_i$ are the end vertices of each path $P_i$. We survey known results giving sufficient conditions for graphs to be $k$-linked, both in general and for specified values of $k$. We also discuss complexity of determining whether a graph is $k$-linked. We further consider weakly $k$-linked graphs, $k$-ordered graphs, and a proper generalization of both $k$-linked and $k$-ordered graphs to $H$-linked graphs.

1. Introduction

Perhaps one of the most important areas of study in graph theory is connectivity and linkage of graphs.

Definition 1.1. A graph is called connected if, for any pair of vertices $u$ and $v$, there exists a path from $u$ to $v$ in the graph.

More interesting, however, than whether a graph is connected, is exactly how connected a graph is. We have the following definition.

Definition 1.2. A graph is called $k$-connected if, for every $X \subseteq V$ with $|X| < k$, $G - X$ is connected.

That is, a graph is $k$-connected if at least $k$ vertices must be removed to render the graph disconnected. An equivalent condition is given by Menger’s Theorem.

Theorem 1.3 (Menger, 1927). Let $G = (V, E)$ be a graph and $A, B \subseteq V$. Then the minimum number of vertices separating $A$ from $B$ in $G$ is equal to the maximum number of disjoint $A - B$ paths in $G$.

This theorem indicates that a graph is $k$-connected if and only if for any disjoint sets $A$ and $B$ of vertices there exist $k$ internally disjoint paths connecting $A$ to $B$. The area of most interest, however, to this paper, asks for a stronger condition. It asks for $k$ disjoint paths connecting $A$ to $B$ where both endpoints of each path are specified.

Definition 1.4. If $G$ is a graph with order $|G| \geq 2k$ and if, for every choice of $2k$ vertices $s_1, \ldots, s_k, t_1, \ldots, t_k$, disjoint $s_i - t_i$ paths $P_i$ exist, then $G$ is called $k$-linked.

Menger’s Theorem makes clear the fact that any $k$-linked graph is also $k$-connected. However, the converse is not true ($C_4$ is 2-connected, but not 2-linked). Results are known, however, which give connectivity conditions guaranteeing that a graph is $k$-linked. We survey these results in the following section.

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2. Connectivity Conditions

We begin with a definition.

**Definition 2.1.** A graph \( G \) is said to contain \( H \) as a minor if \( H \) can be obtained from some subgraph of \( G \) by replacing connected subgraphs to vertices and by contracting paths to edges. \( G \) is said to contain \( H \) as a topological minor or subdivision if \( H \) can be obtained from some subgraph of \( G \) by contracting paths to edges.

The first major result giving a sufficient condition for a graph to be \( k \)-linked is the following, proved independently by Jung in 1970 [9] and by Larman and Mani in 1974 [16].

**Lemma 2.2.** Any \( 2k \)-connected graph containing \( K_{3k} \) as a topological minor is \( k \)-linked.

Taken with the following result of Mader (1967) [18], the existence of a sufficient condition for a graph to be \( k \)-linked dependent only on its connectivity follows.

**Lemma 2.3.** Any graph with sufficiently high average degree contains \( K_m \) as a topological minor.

**Theorem 2.4.** There exists a function \( f(x) \) such that any graph \( G \) that is \( f(k) \)-connected is \( k \)-linked.

The proof of the existence of such a function (hereafter always referred to as \( f(k) \), however, gives only an exponential upper bound on \( f(k) \), that is \( f(k) = O(2^k) \). However, in 1995, Robertson and Seymour [24] improved upon the result of Jung, Larman and Mani.

**Lemma 2.5.** Any \( 2k \)-connected graph containing \( K_{3k} \) as a minor is \( k \)-linked.

Earlier work by Kostochka in 1982 [14] and by Thomason in 1984 [29] placing bounds on the extremal function for \( K_m \) minors in graphs determined that the average degree necessary for the existence of a \( K_m \) minor is of order \( O(m\sqrt{\log m}) \). This gives the following improvement on \( f(k) \).

**Theorem 2.6.** \( f(k) \leq 22k \).

One year later, in 1996, however, this bound was bettered by Bollobás and Thomason [1], who obtained the first linear bound on \( f(k) \) by showing the following:

**Lemma 2.7.** Every \( 2k \)-connected graph with \( 11kn \) edges is \( k \)-linked.

Every \( 22k \)-connected graph must satisfy both of these conditions. Thus we have the following improved bound.

**Theorem 2.8.** \( f(k) \leq 22k \).

In their 2005 paper, Thomas and Wollen [28] proved that every \( 2k \) connected graph with \( 8kn \) edges is \( k \)-linked.

**Theorem 2.9.** \( f(k) \leq 16k \)

Prior to publication of the preceding result, Kawarabayashi, Kostochka, and Yu suggested means by which the result could be improved to a requirement of only \( 6kn \) edges. They themselves improved the result in 2006 [12].

**Theorem 2.10.** \( f(k) \leq 12k \)
Following the suggestion, Thomas and Wollen [28] were able to further improve the bound through a more complicated argument to $5kn$ edges.

**Theorem 2.11.** $f(k) \leq 10k$

To date, this is the best known connectivity condition.

### 3. Other Conditions

#### 3.1. Degree Conditions

The strongest known degree conditions sufficing for a graph to be $k$-linked rely not on the minimum degree of $G$, but on the minimum degree sum of a pair of nonadjacent vertices.

**Definition 3.1.** $\sigma_2(G)$ is defined as the minimum value of $d(u) + d(v)$ over all pairs of nonadjacent vertices $u$ and $v$ in $G$ (where $d(u)$ denotes the degree of $u$).

**Definition 3.2.** $R(n, k)$ is defined as the minimum integer $r$ such that every graph $G$ of order $|G| = n$ satisfying $\sigma_2(G) \geq r$ is $k$-linked.

The following 2006 result is due to Kawarabayashi, Kostochka, and Yu [12].

**Theorem 3.3.**

$$R(n, k) = \begin{cases} 2n - 3 & : n \leq 3k - 1 \\ \left\lfloor \frac{2n+5k}{3} \right\rfloor - 3 & : 3k \leq n \leq 4k - 2 \\ n + 2k - 3 & : 4k - 1 \leq n \end{cases}$$

We make a definition that allows this result to be applied only to the minimum degree of a graph.

**Definition 3.4.** $D(n, k)$ is defined as the minimum integer $d$ such that every graph $G$ of order $|G| = n$ satisfying $\delta(G) \geq d$ is $k$-linked.

The following result is also due to Kawarabayashi, Kostochka, and Yu [12].

**Theorem 3.5.**

$$D(n, k) = \left\lceil \frac{R(n, k)}{2} \right\rceil = \begin{cases} n - 1 & : n \leq 3k - 1 \\ \left\lceil \frac{n+5k}{3} \right\rceil - 1 & : 3k \leq n \leq 4k - 2 \\ \left\lceil \frac{n-3}{2} \right\rceil + k & : 4k - 1 \leq n \end{cases}$$

#### 3.2. Girth Conditions

Recent results allow for considerable weakening of the connectivity conditions if the girth is sufficiently large. In 1991, Mader [19] proved the following theorem.

**Theorem 3.6.** Every $2k$-connected graph with girth sufficiently large is $k$-linked. Furthermore, the condition on the connectivity is sharp. There exist $2k-1$-connected graphs with arbitrarily large girth that are not $k$-linked.

In 2004, Kawarabayashi [11] was able to remove the requirement that the girth be sufficiently large, and to specify exactly what girth will suffice

**Theorem 3.7.** Every $2k$-connected graph with girth at least 11 is $k$-linked if $k$ is not 4 or 5. If $k$ is 4 or 5, girth 19 suffices.
3.3. Minor Conditions. In addition to its use in the proofs of connectivity bounds, conditions on what minors a graph has may be used to classify graphs being \( k \)-linked. As such, we repeat the strongest condition on minors here for the sake of convenience.

**Theorem 3.8** ([24]). *Every \( 2k \)-connected graph containing \( K_{3k} \) as a minor is \( k \)-linked.*


**Theorem 3.9.** *Every \( 2k \)-connected graph containing \( K_{2k,2k} \) as a topological minor is \( k \)-linked.*

4. Results for Specified Values of \( k \)

Besides these general results for \( k \)-linked graphs, many more specific results are known. In 1970, Jung [9] proved the following.

**Theorem 4.1.** *Every \( 4 \)-connected non-planar graph is \( 2 \)-linked.*

Further work in 1980 by Seymour [25], Shiloach [26], and Thomassen [30] resulted in the full classification of graphs that fail to be \( 2 \)-linked, and thus, the full classification of all \( 2 \)-linked graphs.

In their 2005 paper, Chen, Gould, Kawarabayashi, Pfender, and Wei [2] proved several results regarding \( 3 \)-linked graphs.

**Theorem 4.2.** *Every \( 6 \)-connected graph with \( \delta(G) \geq 18 \) is \( 3 \)-linked.*

**Theorem 4.3.** *Every \( 6 \)-connected graph containing \( K_{9} \) as a minor is \( 3 \)-linked.*

In addition, Thomas and Wollen proved the following result.

**Theorem 4.4.** *Every \( 6 \)-connected graph on \( n \) vertices with \( 5n - 14 \) edges is \( 3 \)-linked. The bound on the number of edges is sharp.*

Chen Gould, Kawarabayashi, Pfender, and Wei [2] also described the notion of being \((s_1, s_2, \ldots, s_m)\) linked as the property that for every choice of disjoint vertices of orders \( s_1, s_2, \ldots, s_m \) there exist disjoint connected subgraphs on each of the vertex sets. This generalizes \( k \)-linked, which is simply the case where \( s_1 = s_2 = \cdots = s_k = 2 \). They proved the following result.

**Theorem 4.5.** *Every \( 7 \)-connected graph containing \( K_{9} \) as a minor is \((2,5)\)-linked.*

5. Complexity

The complexity of determining whether a graph is \( k \)-linked varies heavily, depending on whether \( k \) is given or is to be inputted. In his well-known 1975 paper, Karp [10] proved the following.

**Theorem 5.1.** *If \( k \) is part of the input, then determining whether a graph is \( k \)-linked is \( NP \)-complete.*

That same year, Lynch [17] also showed that the problem is \( NP \)-complete even when the graph is planar.

However, when \( k \) is given, the problem is much less complex. Robertson and Seymour proved, in 1990 [22] the following.
Theorem 5.2. For any fixed $k$, a polynomial time algorithm exists to determine whether a graph is $k$-linked.

Further, in the course of three papers [22, 23, 24] Robertson and Seymour gave polynomial time algorithms to determine whether graphs are 2-linked or 3-linked.

6. Weakly $k$-linked Graphs

We turn now to variations on the property of being $k$-linked. An alternate type of connectivity is useful in defining the first variation, a weakening of the $k$-linked property.

Definition 6.1. A graph $G$ is $k$-edge-connected if, for every $X \subseteq E$ with $|X| < k$, $G - X$ is connected.

Definition 6.2. If $G$ is a graph with order $|G| \geq 2$ and if, for every choice of $2k$ vertices $s_1, \ldots, s_k, t_1, \ldots, t_k$, edge-disjoint $s_i - t_i$ paths $P_i$ exist, then $G$ is called weakly $k$-linked.

Every $k$-linked graph is clearly weakly $k$-linked. However, the conditions required for a graph to be weakly $k$-linked are much more easily satisfied. In 1990, Okamura [21] proved the following result.

Theorem 6.3. For $k \geq 2$, every $4k$-edge-connected graph is weakly $3k$-linked, and every $(4k + 2)$-edge-connected graph is weakly $(3k + 2)$-linked.


Theorem 6.4. Every $(2k + 1)$-edge-connected graph is weakly $k$-linked.

Theorem 6.5. Let $G$ be a graph. If $k$ is odd and $G$ is $(k + 1)$-edge-connected, then $G$ is weakly $k$-linked. If $k$ is even and $G$ is $(k + 2)$-edge-connected, then $G$ is weakly $k$-linked.

7. $k$-ordered Graphs

A related property to that of being $k$-linked is that of being $k$-ordered.

Definition 7.1. If $G$ is a graph with order $|G| \geq k$ and if, for every choice of $k$ vertices $s_1, \ldots, s_k$ a cycle exists that encounters the vertices in order, then $G$ is called $k$-ordered. If, in addition, the cycle can always be chosen to be hamiltonian, $G$ is called $k$-ordered hamiltonian.

It turns out that most of the research into $k$-ordered graphs has focused on $k$-ordered hamiltonian graphs. The following conditions are due to Ng and Schultz (1997) [20]. Recall definition 3.1 for $\sigma_2$.

Theorem 7.2. Let $G$ be a graph of order $|G| = n$, and $3 \leq k \leq n$. If $\sigma_2(G) \geq n + 2k - 6$, then $G$ is $k$-ordered hamiltonian.

Theorem 7.3. Let $G$ be a graph of order $|G| = n$, and $3 \leq k \leq n$. If $\delta(G) \geq \frac{n}{2} + k - 3$, then $G$ is $k$-ordered hamiltonian.

Further, the following partial converse also holds [20].

Theorem 7.4. Let $G$ be a hamiltonian graph of order $|G| = n$, $n \geq 3$. If $G$ is $k$-ordered, $3 \leq k \leq n$, then $G$ is $(k - 1)$-connected.
For a more restricted case, Kierstead, Sárközy, and Selkow proved the following result in 1999 [13].

**Theorem 7.5.** If $G$ is a graph with order $|G| = n \geq 11k - 3$ for $k \geq 2$ and if $\delta(G) \geq \left\lceil \frac{n}{2} \right\rceil + \left\lfloor \frac{k}{2} \right\rfloor - 1$, then $G$ is $k$-ordered hamiltonian.

Furthermore, in the same paper, the authors established the relationship between the bound in the preceding theorem and the minimum degree allowable.

**Theorem 7.6.** If $f(k, n)$ is the minimum degree such that every graph $G$ of order $|G| = n$ with $\delta(G) \geq f(k, n)$ is $k$-ordered hamiltonian, and if $g(k, n) = \left\lceil \frac{n}{2} \right\rceil + \left\lfloor \frac{k}{2} \right\rfloor - 1$ then the following relations hold:

\[
\begin{align*}
    f(k, n) &= g(k, n) : 11k - 3 \leq n \\
    f(k, n) &\geq g(k, n) : 2k \leq n \\
    f(k, n) &> g(k, n) : 2k \leq n \leq 3k - 6
\end{align*}
\]


**Theorem 7.7.** If $G$ is a graph of order $|G| = n \geq 53k^2$, $k \geq 3$, and if \(\sigma_2(G) \geq n + \frac{3k^2 - 9}{2}\), then $G$ is $k$-ordered hamiltonian.

In the case where the graph is already known to be $k$-connected and $k$-ordered, they showed that the following stronger result holds.

**Theorem 7.8.** If $G$ is a $k$-connected, $k$-ordered graph of order $|G| = n \geq 8k^2$ and $\sigma_2(G) \geq n$, then $G$ is $k$-ordered hamiltonian.

Two further conditions were given based on the minimum order of the neighborhood of two distinct vertices.

**Theorem 7.9.** If $G$ is a $(k + 1)$-connected, $k$-ordered graph of order $|G| = n \geq 10k$ with the property that $|N(u) \cup N(v)| \geq \frac{n+k}{2}$ for all vertices $u \neq v$, then $G$ is $k$-ordered hamiltonian.

**Theorem 7.10.** If $G$ is a $k$-connected graph of order $|G| = n \geq 18k^2$ with the property that $|N(u) \cup N(v)| \geq \frac{n+k}{2}$ for all vertices $u \neq v$, then $G$ is $k$-ordered hamiltonian.

In 2003, Faudree, Gould, Kostochka, Lesniak, Schiermeyer, and Saito [5] were able to relax the conditions of theorem 7.7 to obtain the following improvement.

**Theorem 7.11.** If $G$ is a graph of order $|G| = n \geq 2k$, $k \geq 3$, and if $\sigma_2(G) \geq n + \frac{3k^2 - 9}{2}$, then $G$ is $k$-ordered hamiltonian.

The authors were also able to use known results to determine minimum degree conditions sufficient for a graph to be $k$-ordered hamiltonian. We define $\delta(n, k)$ in the following manner.

**Definition 7.12.** $\delta(n, k)$ is defined to be the minimum integer $d$ such that every graph $G$ of order $|G| \geq n$ satisfying $\delta(G) \geq d$ is $k$-ordered hamiltonian.
Theorem 7.13.

\[
\delta(n, k) = \begin{cases} 
\left\lfloor \frac{n}{2} \right\rfloor + \left\lceil \frac{k}{2} \right\rceil - 1 & : k \leq \frac{n+3}{11} \\
\frac{n}{2} + \frac{k}{2} - 2 & : \frac{n+3}{11} < k \leq \frac{n}{4} \\
2k - 2 + c & : \frac{n}{4} < k < \frac{2(n+2)}{5} \quad c \geq 0 \\
\left\lfloor \frac{n}{2} + \frac{3k-2}{4} \right\rfloor & : \frac{2(n+2)}{5} \leq k \leq \frac{n}{2} \\
n - 2 & : \frac{n}{2} < k < \frac{2n}{3} \\
n - 1 & : \frac{2n}{3} < k \leq n
\end{cases}
\]

8. H-linked Graphs

The concept of H-linked graphs was introduced independently by Kostochka and Yu in 2005 [15] and by Ferrara, Gould, Tansey, and Whalen in 2006 [6]. It properly generalizes both the k-linked and k-ordered properties.

**Definition 8.1.** Let H be a multigraph, and let G be a graph. G is said to be H-linked if every injective map from V(H) to V(G) can be extended to a topological minor of H where the vertices of H are mapped to their corresponding branch vertices in G.

Intuitively, a graph is H-linked if any ordered set of |H| vertices in G are the branch vertices for a subdivision of H with vertices in that order. If we let H = C_k, then the properties of being H-linked and of being k-ordered are equivalent, so that H-linked generalizes k-ordered. Additionally, if we let H = kK_2, then the properties of being H-linked and of being k-linked are equivalent, so that H-linked generalizes k-linked. Our first result is actually due to Mader in 1991 [19] and predates the introduction of the H-linked property.

**Theorem 8.2.** If H is a graph of size |H| \geq 4, then every 2|H|-connected graph G with sufficiently large girth contains a subdivision with prescribed branch vertices of H.

This naturally implies that G is H-linked. It seems likely that the girth condition can be significantly relaxed, as it was by Kawarabayashi [11] in the k-linked case; however, I am unaware of any such result as yet. More results were given by Kostochka and Yu [15].

**Theorem 8.3.** Let H be a loopless multigraph with k edges and \( \delta(H) \geq 2 \). If G is a graph of order |G| = n \geq 5k + 6, and minimum degree \( \delta(G) \geq \left\lceil \frac{n}{2} + k \right\rceil - 1 \), then G is H-linked. This bound is sharp.

Additionally, if we define \( \eta(H) \) as the maximum size of an edge-cut in H, Ferrara, Gould, Tansey, and Whalen [6] proved the following theorem.

**Theorem 8.4.** If H is a connected multigraph of order |H| = k, and G is a graph of order |G| = n sufficiently large and minimum degree \( \delta(G) \geq \frac{n+n-2}{2} \), then G is H-linked. Furthermore, no more than 2 intermediate vertices need be used to form a path corresponding to an edge of H.

9. Conclusion

Many open questions remain in the study of k-linked graphs. For instance, it is not currently known what the exact connectivity bound function \( f(x) \). It seems
likely from the current work that it can still be improved, or at least improved in some circumstances. It would be interesting to see an exact characterization of that function. In addition, it would be interesting to see what sorts of efficient algorithms could be developed both in general, and for specific types of graphs, or for specific values of $k$.

The notion of an $H$-linked graph is a very recent one and it should be interesting to see what results come forth in the study of this powerful property. It would also be interesting to explore some further extensions of these properties. For instance, just as the $k$-linked property was weakened to the weakly $k$-linked property, the $H$-linked property is being extended to the $H$-immersion property (which requires only edge-disjoint paths, rather than disjoint paths). It seems quite likely that research in this area will be fruitful.

**References**