Generalized Quadrangles:

The generalized quadrangle was first introduced by Jacques Tits in 1959 to describe geometric properties of simple groups of Lie type of rank two. Most of what we know about generalized quadrangles up to 1983, though, is presented in the work done by Stanley Payne and Thas. The name ‘quadrangle’ derives from the fact that four is the smallest number n for which it is possible to draw an n-gon using only lines. Thus, a generalized quadrangle is a generalized polygon of order four.

By definition, a finite generalized quadrangle (GQ) is an incidence structure $S = (P, B, I)$ in which P and B are disjoint (nonempty) sets of objects called points and lines (respectively), and for which I is a symmetric point-line incidence relation satisfying the following axioms:

(I) Each line is incident with $1 + s$ points ($s \geq 1$), and two distinct points are incident with at most one line. That is, there is at most one point on two distinct lines

(II) Each point is incident with $1 + t$ lines, ($t \geq 1$), and two distinct lines are incident with at most one point. That is, there is at most one line through two distinct points

(III) If $P$ is a point and $l$ is a line not incident with $P$, then there is a unique pair $(Q, m) \in P \times B$ for which $P I m I Q I l$. Simply put, for every point $P$, not on the line $l$, there is a unique line $m$ and a unique point $Q$, such that $P$ is on $m$ and $Q$ is on $l$ and $m$.

The integer $s$ and $t$ are called the parameters of the generalized quadrangle, and $S$ is said to have order $(s, t)$. If, though, we have that $s = t$, then $S$ is said to have an order of $s$.

The generalized quadrangle of order $(2,2)$ is the smallest non-trivial example of a generalized quadrangle. It follows that in this quadrangle, each line will contain three points and each point will contain three lines. In addition, there are 15 lines and 15 points in this type of quadrangle. This plane model is known as the “doily,” which is diagramed in the figure below.
There is a point-line duality for GQ (of order \((s, t)\)) for which in any definition or theorem given about generalized quadrangles, we have that the words “point” and “line” are interchanged and, also, the parameters \(s\) and \(t\) are interchanged. Thus, by interchanging points and lines in \(S\), it yields a generalized quadrangle \(S^D\) of order \((t, s)\) and is called the dual of \(S\). Normally, it is assumed that the dual holds in a given theorem or definition of a generalized quadrangle.

We also have that the incidence graph of a generalized quadrangle, which are those with all the lines on a single point, or dually, all the points on a single line. This incidence graph of a generalized quadrangle is connected unless \(P = \emptyset\) or \(L = \emptyset\).

So, from the definition of the finite generalized quadrangle, we can see that if we have two lines that don’t intersect, then we end up obtaining a bunch of quadrangles as you could possibly want, than when in the projective plane we were getting a bunch of triangles that we could ever possibly want. If we obtained any triangles, then we would have the lines intersecting, which cannot happen in a generalized quadrangle.

**Grids and Dual Grids:**

A *grid*, respectively a *dual grid*, is an incidence structure \(S = (P, B, I)\) with

\[
P = \{x_{ij} | i = 0, 1, ..., s_1 \text{ and } j = 0, 1, ..., s_2\}, \quad s_1 > 0 \text{ and } s_2 > 0 \text{ (and dually, we also have,)}
\]

\[
B = \{L_{ij} | i = 0, ..., t_1 \text{ and } j = 0, ..., t_2\}, \quad t_1 > 0 \text{ and } t_2 > 0
\]

\[
B = \{L_0, L_1, ..., L_{s_1}, M_0, M_1, ..., M_{s_2}\}, \quad \text{(and respectively } P = \{x_0, ..., x_{t_1}, y_0, ..., y_{t_2}\}\text{)}
\]

We have that, \(x_{ij} \perp L_k\) if and only if \(i = k\) (respectively, \(L_{ij} \perp x_k\) if and only if \(i = k\)) and \(x_{ij} \perp M_k\) if and only if \(j = k\). (respectively, \(L_{ij} \perp y_k\) if and only if \(j = k\)). The grids, or dual grids, with \(s_1 = s_2\) (and respectively with \(t_1 = t_2\)) are the generalized quadrangles with \(t = 1\) (respectively, \(s = 1\)).

Thus, the above is simply stating that a generalized quadrangle of order \((s, 1)\) is called a *grid* and that of order \((1, t)\) is the *dual grid*
Any ordinary quadrangle is a GQ of order \((1,1)\) and is at the same time a grid and a dual grid, which was the motivation for the term ‘generalized quadrangle.’ Another important note is that a GQ of order \((s, t)\) is called thin if either \(s=1\) or \(t=1\); and in all other cases it is called thick (where \(s > 1\) and \(t > 1\)).

Now, let \(S\) be a generalized quadrangle, a grid, or a dual grid. Given two, not necessarily distinct points \(P, Q\) of \(S\), we write \(P \sim Q\) and say that \(P\) and \(Q\) are collinear, provided that there is some line \(\ell\) for which \(P\) is incident with \(\ell\), and which \(\ell\) is incident with the point \(Q\). Also, we can have that \(P \sim Q\), which means that \(P\) and \(Q\) are not collinear. Dually, for the lines \(\ell, m\) \(\in B\), we write \(\ell \sim m\) or \(\ell \sim m\) according as the lines \(\ell\) and \(m\) are concurrent or noncurrent. If \(P \sim Q\) (or \(\ell \sim m\) we also say that \(P\), or dually \(\ell\), is orthogonal or perpendicular to \(Q\) (or dually \(m\)). The line, which is incident with distinct collinear points \(P, Q\) is denoted by \(PQ\); and the point which is incident with distinct concurrent lines \(\ell, m\) is denoted either by \(\ell m\) or \(\ell \cap m\).

A triad of points is a triple of pairwise noncollinear points. Given a triad \(T = (x, y, z)\), a center of \(T\) is just an element of \(T^\perp\). We say that \(T\) is acentric, centric, or unicentric, according as \(|T^\perp|\) is zero, positive, or equal to one.

Another important note about generalized quadrangles is that the isomorphisms (or collineations), anti-isomorphisms (or correlations), automorphisms, anti-automorphisms, involutions and polarities of generalized quadrangles, grids, and dual grids are defined in the usual way.

**Restriction of the Parameters:**

Let \(S = (P, B, I)\) be a GQ of order \((s, t)\), and put \(|P| = v\) and \(|B| = b\). Then \(S\) has the following properties:

(i) \(v = |P| = (st + 1)(s + 1)\) points

**Proof (i):** Suppose we have a line \(\ell\) such that it will have \(s + 1\) points that are incident with it. Then, there are \(v - (s + 1)\) points, not on \(\ell\). Each point that is not on \(\ell\) is on a unique line meeting \(\ell\). Thus, counting all the points on lines that meet \(\ell\), we can see there are \((s + 1)t\) many lines (because \(\ell\) has \(s + 1\) points and each of these points are incident with \(t + 1\) lines. However, we do not count \(\ell\), so we have \(t\)-many lines). Each of these lines has \(s\) points, not on \(\ell\). So, there is a total of \(st(s + 1)\) points, not on \(\ell\). Thus, \(v - (s + 1) = st(s + 1)\), and so \(v = (s + 1) + st(s + 1) = (st + 1)(s + 1)\)

(ii) \(b = |B| = (st + 1)(t + 1)\) lines

**Proof (ii):** Taking the dual of (i), we get (ii)

(iii) \((s + t)\) divides \(st(1 + s)(1 + t)\)

(iv) If \(s > 1\) and \(t > 1\), then \(t \leq s^2\), and dually \(s \leq t^2\)

(v) If \(s \neq 1, t \neq 1, s \neq t^2,\) and \(t \neq s^2\), then \(t \leq s^2 - s\) and dually \(s \leq t^2 - t\)

**Subquadrangles:**
The GQ $S' = (P', B', I')$ of order $(s', t')$ is a called a subquadrangle, denoted subGQ, of the GQ $S = (P, B, I)$ of order $(s, t)$ if $P' \subset B' \subset B$, and if $I'$ is the restriction of $I$ to $(P' \times B') \cup (B' \times P')$. If $S \neq S'$, then we say that $S'$ is a proper subquadrangle. From $|P| = |P'|$ it follows that $s = s'$ and $t = t'$. So, if $S'$ is a proper subquadrangle then $P \neq P'$, and dually $B \neq B'$.

Let $\ell \in B$. Then only one of the following occurs:

(i) $\ell \in B'$, meaning that $\ell$ belongs to $S'$
(ii) $\ell \notin B'$, and $\ell$ is incident with a unique point $x$ of $P'$, that is that $\ell$ is tangent to $S'$ at the point $P$.
(iii) $\ell \notin B'$, and $\ell$ is incident with no point of $P'$, meaning $\ell$ is external to $S'$. Dually, one could define external points as tangent points of $S'$. From the definition of a generalized quadrangle, it easily follows that no tangent points are incident with a tangent line.

A theorem that is very useful for several characterization theorems of subquadrangles is the following:

**Theorem:** Let $S' = (P', B', I')$ be a substructure of the GQ $S = (P, B, I)$ of order $(s, t)$ for which the following conditions are satisfied:

(i.) If $x, y \in P (x \neq y)$ and $x I \ell I y$, then $\ell \in B'$
(ii.) Each element of $B'$ is incident with $1 + s$ elements of $P'$

Then there are four possibilities:

a.) $S'$ is a dual grid, and so $s = 1$
b.) The elements of $B'$ are lines which are incident with a distinguished point of $P$, and $P'$ consists of those points of $P$ which are incident with these lines
c.) $B' = \Phi$ and $P'$ is a set of pairwise noncollinear points of $P$
d.) $S'$ is a subquadrangle of order $(s, t')$

The Known Generalized Quadrangles and their Properties:

There are three families of generalized quadrangles known as the classical GQs, all of which are associated with the classical groups that were first recognized as generalized quadrangles by Tits. The classical GQs can be used to construct projective planes, inversive planes, Laguerre planes of odd order, and Minkowski planes of even order. The three families of classical GQs are:

i.) A hyperbolic quadric $Q^+(3, q)$, a parabolic quadric $Q(4, q)$, and an elliptic curve $Q^- (5, q)$ are the only possible quadrics in projective spaces over finite fields with a projective index of 1. We find these parameters respectively:

$$Q(3, q) s = q, t = 1, v = (q + 1)^2, b = 2(q + 1)$$

The quadric $Q(3, q)$ is just a grid.
\[ Q(4, q)s = q, t = q, v = (q + 1)(q^2 + 1), b = (q + 1)(q^2 + 1) \]

\[ Q(5, q)s = q, t = q^2, v = (q + 1)(q^3 + 1) \]

ii.) A hermitian \( H(n, q^2) \) has a projective index of 1 if and only if \( n \) is 3 or 4. We find that these parameters are:

\[ H(3, q^2): s = q, t = q, v = (q^2 + 1)(q^3 + 1), b = (q + 1)(q^3 + 1) \]

\[ H(4, q^2): s = q^2, t = q^3, v = (q^2 + 1)(q^5 + 1), b = (q^3 + 1)(q^5 + 1) \]

iii.) A symplectic polarity in \( PG(2d + 1, q) \) has a maximal isotropic subspace of dimension 1 if and only if \( d = 1 \). Here, we find a generalized quadrangle \( W(3,q) \) with \( s = q \) and \( t = q \). The generalized quadrangle derived from \( Q(4,q) \) is always isomorphic with the dual of \( W(3,q) \), and they are both self dual and thus isomorphic to each other if and if \( q \) is even. This has parameters,

**Generalized Quadrangles in Finite Projective Spaces:**

A *projective generalized quadrangle* \( S = (P, B, I) \) is a generalized quadrangle for which the point \( P \) is a subset of the pointset of some projective space \( PG(d,K) \) [of dimension \( d \) over a field \( k \)], and with \( B \) is a set of lines of \( PG(d,K) \). The point \( P \) is the union of all members of \( B \), which are considered as sets of points, and the incidence relation \( I \) is the one induced by \( PG(d,K) \). If \( PG(d',K) \)is the subspace of \( PG(d,K) \) generated by all the points of \( P \), then we say \( PG(d',K) \)is the *ambient space* of \( S \).

All finite projective generalized quadrangles were first determined by Buekenhout and C. Lefevere with a proof most of which is valid in the infinite case. The definition of the generalized quadrangle by Buekenhout and Lefevre was a little more general and included grids. Also, Olanda has given a typically finite proof and Thas and De Winne have given a different combinatorial proof under the assumption that the case \( d = 3 \) is already settled. Recently, though, Dienst has settled the infinite case of the proof of finite projective generalized quadrangles.

**Combinatorial Characterizations on the Known Generalized Quadrangles:**

Several of the theorems of combinatorial characterizations of the known generalized quadrangles are very useful and also important tools in the proofs of certain results concerning strongly regular graphs with strongly regular subconstituents, coding theory, the classification of the antiflag transitive collineation groups of finite projective spaces, the Higman-Sims group, small classical groups, and many more.

Historically, the next result is the oldest combinatorial characterization of a class of GQ. This theorem was discovered independently by several authors:
**Theorem**: A GQ $S$ of order $s$ ($s > 1$) is isomorphic to $W(s)$ if and only if all its points are regular.

Another theorem that arises in the combinatorial characterizations on the known generalized quadrangles is:

**Theorem**: Up to isomorphism there is only one generalized quadrangle of order 2.

**Proof**: Let $S$ be a generalized quadrangle of order 2. Consider two points $x,y$ with $x \sim y$ and let $\{x, y\}^\perp = \{z_1, z_2, z_3\}$. If $\{z_1, z_2\} = \{x, y, u\}$, then by (iv) of the section on the restriction of parameters, we have that $u \sim z_3$. Hence $(x, y)$ is regular. So, every point is regular and $S \cong W(2)$.

**Orders of Known Generalized Quadrangles**:

The orders $(s, t)$ of the known GQs are:

<table>
<thead>
<tr>
<th>$(s, 1)$</th>
<th>$s \in N \setminus {0}$,</th>
</tr>
</thead>
<tbody>
<tr>
<td>$(1, t)$</td>
<td>$t \in N \setminus {0}$,</td>
</tr>
<tr>
<td>$(q, q)$</td>
<td>$q$ any prime power</td>
</tr>
<tr>
<td>$(q, q^2)$</td>
<td>$q$ any prime power</td>
</tr>
<tr>
<td>$(q^2, q)$</td>
<td>$q$ any prime power</td>
</tr>
<tr>
<td>$(q^2, q^3)$</td>
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</tr>
<tr>
<td>$(q^3, q^2)$</td>
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</tr>
<tr>
<td>$(q - 1, q + 1)$</td>
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</tr>
<tr>
<td>$(q + 1, q - 1)$</td>
<td>$q$ any prime power</td>
</tr>
</tbody>
</table>
References:


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