

# CLASSIFICATION OF QUADRICS

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## What is a Quadric?

A quadric is the set of points  $(x_0, x_1, x_2, \dots, x_R)$  of a projective space  $PG(r, F)$  satisfying a homogeneous polynomial  $\phi$  in  $x_0, x_1, x_2, \dots, x_R$ , given by

$$\phi = \sum_{0 \leq i, j \leq n} M_{ij} v_i v_j \quad (1)$$

where  $(v_0, \dots, v_n)$  are the coordinates of  $v$  with respect to some chosen basis, and  $M$  is a certain  $(n+1) \times (n+1)$  symmetric matrix with entries  $M_{ij}$  in a scalar field  $K$ , that depends on  $F$  and on the basis. Dually, a quadric is the set of hyper-planes of  $PG(r, F)$  that satisfies a homogeneous polynomial of degree two, called a *quadric envelope*.

## Classification of Quadrics

A quadric is classified as being *singular* or *non-singular*, *degenerate* or *non-degenerate*, *reducible* or *irreducible*, and *ruled* or *non-ruled*. Quadrics which have the same classification are projectively equivalent. There are three distinct types of quadric in projective geometry, distinct because no real projective transformation can transform a member of one type into a member of another. These types are:

- The Ellipsoid and its projective equivalents
- The Hyperbolic paraboloid and its projective equivalents
- The Cone and its projective equivalents

## Definitions:

**Singular:** If a quadric contains at least one point  $P$  such that every line through  $P$  intersects the quadric doubly there,  $P$  is called a singular point and the quadric is singular. **Non-singular** quadrics have no singular points.

**Degenerate:** Degenerate quadrics have singular points, **non-degenerate** quadrics have non-singular points.

**Reducible:** A quadric is **reducible** if  $\phi = 0$  splits into two linear equations in  $F$ . Otherwise, it is irreducible.

**Ruled:** A surface  $S$  is **ruled** if there exist families of parallel lines on  $S$ . The most familiar examples are the plane and the curved surface of a cylinder or cone. A surface is **doubly ruled** if through every one of its points there are two distinct lines that lie on the surface. The hyperbolic paraboloid and the hyperboloid of one sheet are doubly ruled surfaces.

## Equation of a Quadric

In  $PG(3,F)$ , the homogeneous equation described by (1) may be written as:

$$\Phi(x,y,z,t) = ax^2 + by^2 + cz^2 + dt^2 + fyz + gxz + hxy + lxt + myt + nzt = 0 \quad (2)$$

This equation has 10 terms, in accordance with Theorem 7.3 of the text (p. 133). Thus, a quadric has  $10 - 1 = 9$  degrees of freedom (DOF) in  $PG(3,F)$ , which means that in  $PG(3,F)$ , 9 points are sufficient to determine a quadric.

Note: All of the terms in the equation describing a quadric are quadratic. Example:  $x^2 + y^2 + 2yz = 0$  is the equation of quadric,  $x^2 + y^2 + 2y = 0$  is not the equation of a quadric since  $2y$  is linear, not quadratic.

## Matrix Associated with a Quadric

Taking the partial derivatives of equation (2) with respect to  $x, y, z,$  and  $t$  and arranging them in a matrix yields:

$$M = \begin{bmatrix} 2a & h & g & l \\ h & 2b & f & m \\ g & f & 2c & n \\ l & m & n & 2d \end{bmatrix} = 0$$

Let  $Q$  be a quadric.  $M$  is called the matrix form associated with the quadric  $Q$ .  $M$  is always an  $(n+1) \times (n+1)$  *symmetric* matrix. It is common to write  $Q = MV = 0$  as  $Q = x^T Ax = 0$  where  $M = 2A$ :

$$A = \begin{bmatrix} a & \frac{h}{2} & \frac{g}{2} & \frac{l}{2} \\ \frac{h}{2} & b & \frac{f}{2} & \frac{m}{2} \\ \frac{g}{2} & \frac{f}{2} & c & \frac{n}{2} \\ \frac{l}{2} & \frac{m}{2} & \frac{n}{2} & d \end{bmatrix} = 0$$

Four cases may occur when solving  $MV=0$ , where  $V$  is a column vector whose elements are  $x,y,z,$  and  $t$ :

**Case 1:** Rank  $M = 4$ . Trivial solution is the only solution:  $x=y=z=t=0$ . Quadric is non-singular.

Example 8.14:  $\phi = xt - yz = 0$  is a quadric in  $PG(3,F)$ . The partial derivatives, when put in matrix form and set equal to zero, imply that  $x=y=z=t=0$ , which is the trivial solution, and so the quadric is non-singular.

**Case 2:** Rank  $M = 3$ . There is a unique point  $V$  which satisfies  $MX = 0$ , and lies on  $Q$ .

Exercise 8.1.1:  $\phi = x^2 + 2yz + 2zx + 2xy = 0$  is a quadric in  $PG(3,F)$ . Take partial derivatives with respect to  $x, y,$  and  $z,$  and set in matrix form. After row reducing, it is found that matrix  $M$  has rank 3, which means there is a singular point lying on the quadric. This point is the vertex of a **quadric cone**, and has homogeneous coordinates  $V = (0,0,0,1)$ ,

corresponding to the plane  $t=0$ . It can be shown that the point  $V$  is *always*  $(0,0,0,1)$ , corresponding to the plane  $t = 0$ .

**Case 3:** Rank  $M = 2$  and there exists a line  $m$  of  $Q$  such that every point of  $m$  is a singular point of  $Q$ . Plane pairs are generated.

This is the case when two of  $x, y, z,$  or  $t$  are squared.  $\Phi$  takes the form  $x^2 + y^2 = 0$ . This has algebraic solution  $x = \pm iy$ , which means there are conjugate points in the complex plane as solutions. Dually, in the projective space, there are two planes that are not coincident.

**Case 4:** Rank  $M = 1$ .  $Q$  splits into two coinciding planes. If rank  $M = 1$ , there is only 1 linearly independent variable. Three of  $x, y, z,$  and  $t$  are 0. Two coincident planes are generated.

This is the case when only one of  $x, y, z,$  or  $t$  is squared. Since  $x^2 = 0$  has algebraic solution  $x = 0$ ,  $x = 0$ , there are two coincident points on the same line. Dually, in the projective space, there are two coincident planes in the same space.

### Classification through Signature

Any given quadric of a certain type can be translated and rotated so that it is congruent to another quadric of the same type. This means that the homogeneous equations of the quadric can be fit into a canonical form. Canonical forms are found only when the matrix  $M$  (or  $A$ ) is diagonal. There is a theorem in linear algebra that says any symmetric matrix can be diagonalized through the use of a suitable linear transformation. Since both of the above matrices are symmetric, they can be diagonalized. Diagonalizing  $A$  and using the form  $Q = x^T A x = 0$ , we obtain the following equation:

$$ax^2 + by^2 + cz^2 + dt^2 = 0$$

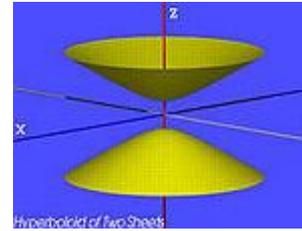
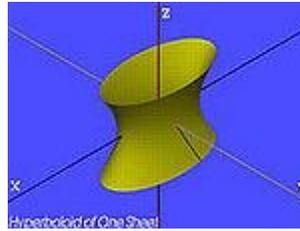
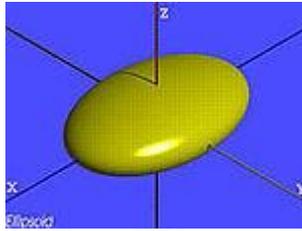
Normalizing this equation by dividing by  $t^2$  we end up with:

$$ax^2 + by^2 + cz^2 + d = 0$$

where  $t^2$  has been incorporated into the remaining coefficients and  $d$  is any scalar constant. We are free to choose what value we give to  $d$ . Let us assign  $d = 1$ . By varying the signs of each of the coefficients and setting  $d = 0$  or  $1$ , we obtain the equations of the quadric.

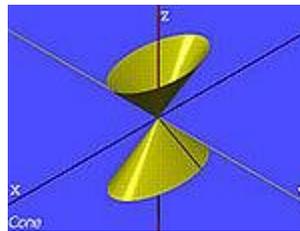
**Case 1a:**  $M$  has rank 3 and  $d \neq 0$ . The possible combinations for the signs are:  $(+++)$ ,  $(++-)$ ,  $(+-)$ , and  $(---)$ . The following equations are obtained:

- $x^2 + y^2 + z^2 - 1 = 0$ , an ellipsoid ( $E$ )
- $x^2 + y^2 - z^2 - 1 = 0$ , a one-sheet hyperboloid ( $H1$ )
- $x^2 - y^2 - z^2 - 1 = 0$ , a two-sheet hyperboloid ( $H2$ )
- $-x^2 - y^2 - z^2 - 1 = 0$ , an empty quadric, called an imaginary ellipsoid ( $IE$ ), not shown



**Case 1b:** M has rank 3 and  $d = 0$ . Possible sign combinations are:  $(+++)$ ,  $(++-)$ . The following equations are obtained:

- $x^2 + y^2 + z^2 = 0$ , an imaginary cone with vertex at the origin
- $x^2 + y^2 - z^2 = 0$ , a real cone with vertex at the origin



**Case 2a:** M has rank 2 and  $d = -1$ . The equations of three *cylinders* are obtained:

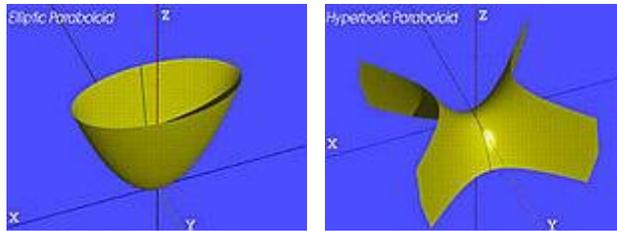
- $x^2 + y^2 - 1 = 0$ , an imaginary cylinder
- $x^2 - y^2 - 1 = 0$ , a hyperbolic cylinder
- $-x^2 - y^2 - 1 = 0$ , an elliptic cylinder

**Case 2b:** M has rank 2 and  $d = 0$ . Equations of planes are generated:

- $x^2 + y^2 = 0$ , a pair of complex conjugate planes containing the z-axis
- $x^2 - y^2 = 0$ , a pair of real planes containing the z-axis

**Case 2c:** M has rank 2. Possible combinations are  $(++)$  and  $(+-)$ . The following equations are obtained:

- $x^2 + y^2 - d = 0$ , an *elliptic paraboloid (EP)*
- $x^2 - y^2 - d = 0$ , a *hyperbolic paraboloid (HP)*



**Case 3a:** M has rank 1. One equation is generated,  $x^2 + d = 0$ , with  $d = -1, 0, 1$ . Equations are:

$x^2 - d = 0$ , *two complex conjugate planes*

$x^2 = 0$ , *two real coincident planes*

$x^2 + d = 0$ , *two real parallel planes*

**Case 3b:** M has rank 1. One equation is generated:

$x^2 + 2dz = 0$ , giving a *parabolic cylinder (PC)*

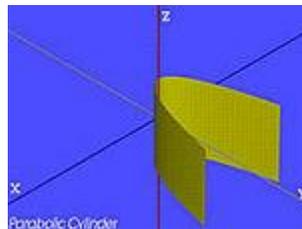


Table 1 lists the quadric types and their classifications.

**Table 1: Quadric Types and Their Classifications**

Quadric Type	Singular	Degenerate	Ruled	Reducible
Ellipsoid (E)	No	No	No	No
Hyperboloid of 2 Sheets (H2)	No	No	No	No
Imaginary Ellipsoid (IE)	No	No	No	No
Elliptic Paraboloid (EP)	No	No	No	No
Parabolic Cylinder (PC)	No	No	No	No
Imaginary Cone	Yes	Yes	Yes	No
Real Cone	Yes	Yes	Yes	No
Imaginary Cylinder	Yes	Yes	Yes	No
Hyperbolic Cylinder	Yes	Yes	Yes	No
Elliptic Cylinder	Yes	Yes	Yes	No
Hyperbolic Paraboloid (HP)	No	No	Yes (Doubly)	No
Hyperboloid of 1 Sheet (H1)	No	No	Yes (Doubly)	No
Pair of Complex Conjugate Planes Containing the z-axis	Yes	Yes	Yes (Triply)	Yes
Pair of Real Planes Containing the z-axis	Yes	Yes	Yes (Triply)	Yes
Real Double Plane	Yes	Yes	Yes (Triply)	Yes
Real Parallel Planes	Yes	Yes	Yes (Triply)	Yes
Complex Parallel Planes	Yes	Yes	Yes (Triply)	Yes

From the table, it is seen that E, H2, EP and IE are projectively equivalent, the cones and the cylinders are projectively equivalent, and HP and H1 are projectively equivalent.

## REFERENCES

1. Rey Casse, *Projective Geometry an Introduction*, Oxford University Press, 2006
2. Harry Levy, *Projective and Related Geometries*, The MacMillan Company, New York, 1964
3. Pierre Samuel, *Projective Geometry (translated from Geometrie projective, Presses Universitaires De France, by Silvio Levy)*, Springer-Verlag New York Inc., 1988

Helpful Websites:

[http://viswiki.com/en/Quadric \(projective geometry\)](http://viswiki.com/en/Quadric (projective geometry))

<http://www.nct.anth.org.uk/algebrai.htm>

[http://en.wikipedia.org/wiki/Quadric \(projective geometry\)](http://en.wikipedia.org/wiki/Quadric (projective geometry))

[http://www.usd.edu/~jflores/MultiCalc02/WebBook/Chapter\\_13/Graphics/Chapter13\\_6/Demo\\_Html13\\_6/13.6%20Cylinders%20and%20Quadric%20Surfaces.htm](http://www.usd.edu/~jflores/MultiCalc02/WebBook/Chapter_13/Graphics/Chapter13_6/Demo_Html13_6/13.6%20Cylinders%20and%20Quadric%20Surfaces.htm)

<http://mathworld.wolfram.com/QuadraticSurface.html>