Inversive geometry comes from a certain fundamental theorem based on inversions.

**Definition 1.** *Inversions* are the study of transformations of the Euclidean Plane.

**Theorem 1.** The *Fundamental Theorem of Inversive Geometry* states that by letting \( P, Q, R \) and \( P', Q', R' \) be two sets of three points in a plane, there exists an inversive transformation that maps \( P \mapsto P' \), \( Q \mapsto Q' \), and \( R \mapsto R' \).

These transformations preserve angles and map generalized circles into generalized circles, where generalized circle means either a circle or a line. Circle inversion, specifically, gives the information needed to understand the inversive plane of circles. By using the inversive version of the Euclidean plane, circles and lines fit under one rubric—inversive circles (P. Beem).

The inverse of a point \( P \) with respect to a (reference) circle of center \( O \) and radius \( r \) is a point \( P' \) such that \( P \) and \( P' \) are on the same ray going from \( O \), and \( OP \times OP' = r^2 \).

Equation for inversion in 2D: \( OP \times OP' = r^2 \) (S. Phelps).

The transformation of the plane which takes each point \( P \) to its inverse \( P' \) is called the inversion relative to the given point. Note that this inversion takes \( P' \) back to \( P \) so the transformation obtained applying the same inversion twice is simply the identity transformation. The inverse of a point inside the reference circle is outside the reference circle and vice-versa. Also note that a point on the circle stays in the same place under inversion. In general, the closer a point is to the center of the circle, the further away its transformation is.

**The Inversion Theorem (2):** Let \( C \) be a circle with center \( O \). For any object \( X \), let \( X' \) denote the inverse of \( X \) with respect to \( C \).
1. If $L$ is an extended line through $O$, then $L \plane L$.
2. If $L$ is an extended line not through $O$, then $L \plane$ is a circle through $O$.
3. If $D$ is a circle through $O$, then $D \plane$ is an extended line through $O$.
4. If $D$ is a circle not through $O$, then $D \plane$ is a circle not through $O$.

**Theorem 3.** The point $P \plane$ is inverse to point $P$ if and only if $P$ is inverse of $P \plane$.

**Theorem 4.** The inversion maps every point outside the circle to some point inside the circle and vise-versa. Each point on the circle of inversion is mapped to itself (T. Lewis).

Also note that two lines that meet at a single point in the Euclidean plane meet at two points in the inversive plane. Two lines that are parallel in the Euclidean plane meet only at the ideal point in the inversive plane. (T. Lewis).

Inversion under 3 dimensions is generalized to sphere inversion, that is, inversion of a point $P$ with respect to a reference sphere centered at $O$ with radius $r$ is a point $P \plane$ such that $OP\times OP' = r^2$, and $P, P'$ are on the same ray from $O$.

Equation of inversion in 3D: $OP \times OP' = r^2$

This transformation in circle was invented by L.I. Magnus in 1831. The following are 6 helpful properties in understanding inversive transformations, and may overlap slightly with previously stated information:

1. Every inversion needs a circle of inversion and a center of inversion (the center of the circle).
2. The circle of inversion is fixed, that is, every point on the circle of inversion is mapped to itself under inversion.
3. Inversion preserves angles; more specifically, perpendicularity is preserved by inverse transformations.
4. Circles orthogonal to the circle of inversion and lines through the center of inversion (which are orthogonal to the inversion circle as well) are invariants under inversion (meaning their inverses of themselves).
5. The image of a circle under inversion depends on whether or not the circle goes through the center of inversion. If it does, the image will be a line. Otherwise, the image will be a circle.

6. The image of a line under inversion depends on whether or not the line goes through the center of inversion. If it does, the image is a line. Otherwise, it will be a circle. (S. Phelps).

Let it be noted, then, that an inversive circle is an ordinary circle if and only if it doesn’t contain the ideal point. If it does, it then becomes an ordinary Euclidean line.

**Theorem 5.** Let $C$ be a circle and let $P$ and $P'$ be an inverse pair of points with respect to $C$. Then every circle through $P$ and $P'$ is orthogonal to $C$.

Given a circle $C$ in plane $W$, every point $P$ in the plane has an inverse with respect to $C$ except the center of the circle, $O$. The point $O$ has no inverse and isn’t the inverse of any other point. Point $O$, therefore, is mapped to the point at infinity $I$. This brings us to the definition of inversive planes: the Euclidean Plane together with this single ideal point at infinity is called the inversive plane (T. Lewis). They are also known as “Mobius” planes (S. Payne). An inversive plane is a geometry with three undefined notions: points, circles, and an incidence relation between points and circles satisfying the following three axioms:

1. Through any 3 distinct points there is exactly one circle.
2. If $P, Q$ are points and $C$ a circle passing through $P$ but not through $Q$, then there is a unique circle $C'$ passing through $Q$ such that $C \cap C' = \{P\}$.
3. There exists 4 points which do not all lie on a common circle.

(University of Wyoming).

**Theorem 6.** Each finite inversive plane of even order is egglike. Dembowski [1964].

Inversive plane of order $n$ is a $3-(n^2+1, n+1, 1)$ design and is categorized into two classes, “points” or “circles” which are also known as blocks (E. Bert). Blocks exist in the form of an $n (n^2 + 1)$ design. An inversive plane of order $n$ has $(n^2+1)$ points and $n (n^2 + 1)$ circles. Every circle is incident with $(n+1)$ points and any two points are incident with $(n+1)$ circles (Ball).

**Theorem 7.** For any point $P$ of an inversive plane, the points not equal to $P$ and the circles incident with $P$ form an affine plane (S. Ball).
Two circles can behave in the following ways: disjoint, tangent, secant (S. Payne).

**Definition 2.** Two circles are said to be disjoint if they have no common points.

![Disjoint Circles](image)

**Definition 3.** Two circles are said to be tangent if they have one common point.

![Tangent Circles](image)

**Definition 4.** Two circles are said to be secant if they have two common points (S. Payne).

![Secant Circles](image)

**Theorem 8.** Given a circle $C$, a point $P$ on $C$ and a point $Q$ not on $C$, there exists a unique circle $D$ containing $Q$ which is tangent to $C$ at the point $P$ (E. Bert).

All known examples of finite inversive planes come from one of two infinite families, both of which are “egglike” in the sense that they arise by taking non-tangent planar sections of an ovoid in projective 3-space over a finite field (E. Bert).

**Definition 5.** An inversive plane arising from an ovoid is “egglike” (J. Thas).

**Theorem 9.** Each finite inversive plane of even order is egglike (J. Thas).

Many designs arise under the inversion of a circle. We have what are called “pencils” of circles, “flocks” of circles, and “bundles” of circles (S. Payne).
**Definition 6.** A *pencil of circles* is a maximal collection of mutually tangent circles through a common point. The common point is termed “carrier” of the pencil (S. Payne). For finite inversive planes of order $n$, each pencil has $n$ circles.

There are $n(n+1)$ circles through $P = (n+1)$ pencils each with carrier $P$ (S. Ball).

**Definition 7.** An *elliptic pencil* of inversive circles is that of a maximal collection of inversive circles that meet in two inversive points (P. Beem). Every two circles of an elliptic pencil pass through the same two points. This does not include imaginary circles.

**Definition 8.** A *hyperbolic pencil* of inversive circles is a maximal collection of circles, no two of which intersect (P. Beem). It includes real and imaginary circles.

Under inversion, circles transform into circles, as well as pencils into pencil. This includes parabolic pencils into parabolic pencils, elliptic pencils into elliptic pencils, and hyperbolic pencils into hyperbolic pencils.

**Theorem 10.** For every point other than the common point, there is a unique member of the pencil through it (P. Beem).
Theorem 11. For any pencil of inversive circles, there is a complementary pencil consisting of all the inversive circles that are orthogonal to all the members of the original pencil (P. Beem).

Any pencil of inversive circles contains exactly one straight line called the radical axis of the pencil (P. Beem).

Definition 9. A flock of circles is a set of pairwise disjoint circles such that, with the exception of 2 points $P$ and $Q$, every other point lies on exactly one circle in the flock. The two points are termed “carriers” of the flock (S. Payne).

Definition 10. A bundle of circles is the set of all circles passing through both of a given pair of distinct points, $P$ and $Q$ (S. Payne). This would look similar to that of an elliptic pencil design.

Under circle inversion, we have Pappus’ Configuration for circles. His configuration states the following: Let $A$, $B$, $C$, $D$, $E$, and $F$ be any six point in the inversive plane. Draw circles $ACE$ and $BDF$. Choose a point $A'$ on the circle $ACE$. Draw the circle $ABA'$ and let $B'$ be the point besides $B$ where the circle intersects the circle $BDF$. Draw the circle $BCB'$ and let $C'$ be the other point where it intersects $ACE$. Draw the circle $CDC'$ and let $D'$ be the other point where it intersects $ACE$. Draw the circle $DED'$ and let $E'$ be the other point where it intersects $BDF$. Draw the circle $FAF'$, and let $F'$ be the other point where it intersects $BDF$. Draw the circle $FAF'$. Then $A'$ lives on the circle $FAF'$ (D. Joyce).

Theorem 12. Pappus’ Theorem under Circles says let $X$ and $X'$ be the points where the circles $ABA'$ and $DED'$ meet, if they do, and $Y$ and $Y'$ where $BCB'$ and $EFE'$ meet, and $Z$ and $Z'$ where $CDC'$ and $DED'$ meet. Then $X$, $X'$, $Y$, $Y'$, $Z$, $Z'$ are concyclic (D. Joyce).

Definition 11. A concyclic circle is when there exists a circle passing through all the points in the set.
Pappus’ Configuration for Circles (D. Joyce).

References


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