Laguerre Planes: A Basic Introduction

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1 Introduction

Like a projective plane, a Laguerre plane is a type of incidence structure, defined in terms of sets of elements and an incidence relationship between them. The sets of a Laguerre plane are generally called points and circles, with the circles being set of points. The incidence relationship and other properties will be discussed in Section 2 below, along with some simple theorems.

The intent of this paper is to give a brief overview of the definition and significance of Laguerre planes, so after the basics are laid out in Section 2, Section 3 gives the classical (and simplest) model of an infinite Laguerre plane, which uses a cylinder. Section 4 then shows how an octahedron can model a small finite Laguerre plane. Section 5 discusses what I believe was Laguerre's own model for a Laguerre plane, which is more complicated and challenging than the previous two models, but too interesting not to include. Finally, Section 6 will briefly discuss the relationship between Laguerre planes and other areas of geometry.

2 Definition and a Few Theorems

The following definition of a Laguerre plane is adapted from Schroth[8]. For interestingly different but roughly equivalent definitions, see [7], [3], or [5].

Consider an incidence structure \((P, C, I)\) with points \(P\) and circles \(C\) and an incidence relation \(I\) containing point-circle pairs \((P, c)\) where \(P\) is contained in \(c\).

**Definition 1.** Two points are parallel if they are the same point, or if they have no circle in common. Otherwise, they are non-parallel.

**Definition 2.** Two circles touch if they are the same circle, or if they have exactly one point in common. If two circles have only point \(P\) in common, they touch in \(P\).

**Definition 3.** Such an incidence structure is a Laguerre plane if it satisfies the following axioms:

**Axiom 1.** Circle Generation: Three pairwise non-parallel points can be joined by a unique circle.

**Axiom 2.** Touching: Given non-parallel points \(P\) and \(Q\) and circle \(c\) containing \(P\), there is exactly one circle \(d\) through \(Q\) touching \(c\) in \(P\).

**Axiom 3.** Parallel Intersection: Given a point \(P\) and a circle \(c\), there is a unique point on \(c\) parallel to \(P\).

**Axiom 4.** Richness: There is a circle containing at least three but not all points.

One unusual feature of a Laguerre plane is the notion of points being parallel or non-parallel. This is a defining feature of circle planes, discussed in Section 6, of which Laguerre planes are a subset.

Here are a few basic theorems that follow directly from the axioms.
Theorem 1. Parallelism between points is an equivalence relation.

Proof. Parallelism satisfies the three requirements of an equivalence relation – symmetry, reflexivity, and transitivity – as follows.

As stated in Axiom 1, points are parallel when they are coincide, i.e., a point is parallel to itself, so parallelism is reflexive.

Given that distinct points are parallel when they do not share a common circle, it is clear that the relation is symmetrical, i.e., if \( P \) is parallel to \( Q \), then \( Q \) is parallel to \( P \).

We show transitivity by contradiction. Choose points \( P, Q, \) and \( R \) such that \( P \) is parallel to \( Q \) and \( Q \) is parallel to \( R \). If parallelism is transitive, then \( P \) is also parallel to \( R \). Suppose \( P \) and \( R \) are not parallel. Then \( P \) and \( R \) must lie in some common circle \( c \). Yet, by Axiom 3 (Parallel Intersection), \( c \) can contain only one point parallel to \( Q \), a contradiction. Thus \( P \) must be parallel to \( R \) and parallelism is transitive.

The following two theorems are given as properties in [2].

Theorem 2. Any two distinct circles have at most two points in common.

Proof. Since any three non-parallel points form a unique circle, it is impossible for two distinct circles to have three or more points in common.

Theorem 3. If circles \( c \) and \( d \) touch in point \( P \), and if \( d \) also touches circle \( e \) in \( P \), then \( c \) and \( e \) touch in \( P \).

Proof. Assume \( c \) and \( d \) touch in point \( P \) (i.e., \( c \cap d = \{ P \} \)), and that \( d \) and \( e \) also touch in \( P \), but that \( c \) and \( e \) do not touch in \( P \). Since \( c \) and \( e \) do both contain \( P \), in order for them not to touch in \( P \), they must share another common point, \( Q \). If \( d \) contained \( Q \) it would have two points in common with \( c \) and \( e \) and would not touch either \( c \) or \( e \). So, given that \( P \) and \( Q \) are two non-parallel points, with \( P \) in \( d \), by Axiom 2 (Touching), there is a unique circle through \( Q \) touching \( d \) in \( P \). However, both \( c \) and \( e \) are circles through \( Q \) touching \( d \) in \( P \), a contradiction. Therefore, \( c \) and \( e \) do not have a second point in common, and touch in \( P \).

This theorem establishes that touching in a particular point is transitive.

With this definition and these properties in mind, we now introduce three models of a Laguerre plane.

3 The Classical Cylinder Model of a Laguerre Plane

Perhaps the most straightforward model of a Laguerre plane uses the unit cylinder \( S^1 \times \mathbb{R} \) in \( \mathbb{R}^3 \). \( \mathcal{P} \) consists of the points on this cylinder, and \( \mathcal{C} \) consists of the circles and ovals formed by intersections of the cylinder with any non-vertical planes. Points are incident with circles in the standard way.
In this model, points are parallel if they are on the same vertical line, as will be shown.

![Figure 1: Points are parallel if contained in the same vertical line. Points on AB and PQ thus constitute two different parallel classes. Circle m intersects each class, or vertical line, in a unique point.](image)

This model satisfies the axioms above as follows:

1 *Circle Generation.* Take three points in \( P \). Since these points are in \( \mathbb{R}^3 \), they determine a unique plane \( \pi \). \( \pi \) will be non-vertical if and only if no two of the points are on the same vertical line. Therefore, if and only if no two of the points are on a vertical line, the intersection of \( \pi \) with the unit cylinder is the unique circle determined by these points. Hence, points on the same vertical line are parallel, points not on a vertical line are non-parallel, and three pairwise non-parallel points determine a unique circle.

2 *Touching.* Choose two non-parallel points \( P \) and \( Q \) and a circle \( c \) through \( P \).

There are two cases:

- **Case 1** \( Q \) is on \( c \). In this case, \( c \) is the required circle. Any other circle through both \( P \) and \( Q \) would intersect \( c \) in both of these points, and therefore not touch \( c \) (which requires a single point in common).

- **Case 2** \( Q \) is not on \( c \). Let \( \gamma \) be the plane containing \( c \). Since \( P \) is on \( c \), \( P \) is also contained in \( \gamma \). Let \( l \) be the unique (Euclidean) line in \( \gamma \) tangent to \( c \) through \( P \). \( Q \) is not in \( \gamma \). Let the unique plane containing both \( Q \) and \( l \) be called \( \delta \). \( \delta \) is non-vertical because \( PQ \) is not vertical and \( l \) is not vertical. Let \( d \) equal the intersection of \( \delta \) with the unit cylinder. Now \( d \) is a circle in our model. Since \( \delta \cap \gamma = \{ l \} \) and \( l \) is tangent to \( c \), \( d \cap c = \{ P \} \). Thus, \( d \) is a circle through \( Q \) that touches \( c \) at \( P \).
Let $d'$ be some other circle through $Q$ touching $c$ in $P$. Then $d'$ lies in some plane, $\delta'$, that contains $P$. $\delta'$ intersects $\gamma$ in a line; this line must contain $P$. Since $d'$ only intersects $c$ in $P$, the line of intersection must be $l$, the unique line through $P$ tangent to $c$ in $\gamma$. $\delta'$ thus contains both $l$ and $Q$, and is therefore identical to $\delta$, so that $d'$ is identical to $d$. Therefore, the desired circle is unique.

3 Parallel Intersection. Axiom 3 is equivalent to saying that each circle contains exactly one point of each parallel class, or vertical line. Let $l$ be one such vertical line, and $c$ a circle in our model. There is a unique non-vertical plane $\gamma$ of which $c$ is a subset. Since $l$ is neither contained in, nor parallel (in the Euclidean sense) to, $\gamma$, their intersection is a unique point which constitutes the unique point of $l$ contained in $c$.

4 Richness. Our model has an infinite number of points in each circle, and no circle contains all of the points.

A similar model may be obtained using a cone with vertex removed (see, for instance, [1] or Chapter 14 of [5]).

4 A Finite Model Using an Octahedron

In a finite Laguerre plane, every circle contains $n + 1$ points for some $n$; $n$ is the order of the plane. Also, every parallel class contains $n$ points. Like the Fano projective plane, the following is a model of the Laguerre plane of order 2 – the smallest Laguerre plane. This model comes from Burkard Polster’s nifty Geometrical Picture Book [6].

Let $ABCDEF$ be an octahedron in $\mathbb{R}^3$, as shown in Figure 2. Let the points be the six vertices \{A, B, C, D, E, F\}, and let the circles be the eight faces of the octahedron: $ABC$, $AED$, and so on.

![Octahedron ABCDEF](image)

Figure 2: Octahedron $ABCDEF$. 
From Definition 1, it follows that the points that are parallel - that is, that do not have a circle in common - are the opposing points of the octahedron. $A$ is parallel to $F$, $B$ to $D$, and $C$ to $E$. (Also recall, of course, that each point is trivially parallel to itself.) Since circles touch if and only if they contain exactly one common point, the circles that touch here are those that are across from each other, e.g., $AED$ and $ABC$.

This model satisfies the axioms as follows:

1 Circle Generation. Aside from the opposing points, which are parallel, any three points create a unique face of the octahedron.

2 Touching. Since this model is isomorphic to all of its rotations, it will suffice to prove this for one pair of non-parallel points. Let us choose $A$ and $B$. We hope to show that given a circle (face) $c$ containing $A$, we can find a unique circle through $B$ that touches $c$ in $A$. The circles containing $A$ are $ABC$, $ACD$, $ADE$, and $ABE$. It is easy to confirm by inspection that the unique circle through $B$ that touches $ABC$ in $A$ is $ABC$ itself; for $ACD$, we get $ABE$; for $ADE$, $ABC$; and for $ABE$, $ABE$.

3 Parallel Intersection. It can be verified by inspection that every face of the octahedron contains either $A$ or $F$, either $B$ or $D$, and either $C$ or $E$.

4 Richness. $ABC$ is a circle containing at least three but not all points.

Given that a hexahedron (e.g., a cube) is the dual of an octahedron (that is, they are equivalent if faces are swapped with vertices), an equivalent Laguerre plane may be formed by taking the sides of the hexahedron as the points and the vertices as the circles.

5 Laguerre’s Model

The following description of a Laguerre Plane is given by Robert Knight[2] and is based on Laguerre’s work in *Sur La Géométrie de Direction*[4]. It is rather baroque compared to the previous two models.

In the Euclidean plane, we define a spear as an oriented line, so that each Euclidean line underlies two spears pointing in opposite directions. (For instance, $y = x$ underlies a spear pointing “northeast” and another pointing “southwest”). A cycle is either a single point (a point cycle) or an oriented circle (a proper cycle) that works like the oriented line, with each Euclidean circle underlying clockwise- and counter-clockwise cycles.

In this model, $P$ consists of the spears, and $C$ consists of the cycles (both point and proper). A spear is incident to a point cycle in the usual way (i.e., if the point is in the spear) and to a proper cycle if it is tangent to the underlying circle with agreeing orientation at the point of tangency.

Spears are parallel if and only if their underlying Euclidean lines are parallel and they have the same orientation; in this situation, no cycle can touch both. (See Figure 4.) It also follows in this model that cycles touch if and only if they
are tangent with agreeing orientation at the point of tangency; otherwise, it is not possible for them to share exactly one spear. (See Figure 5.)

Figure 4: \( S \) and \( T \) are parallel, and the circle underlying cycle \( c \) is tangent to both. However, \( c \) touches only \( S \); its orientation does not agree with \( T \)'s at the point of tangency.

Showing that this model satisfies the axioms is slightly more complicated than for the previous two models. Informally, it goes as follows:

1 \textit{Circle Generation}. We are given three spears, no two of which are parallel. There are three cases to consider here. Figure 6 illustrates these possibilities.

\textbf{Case 1} If the three spears are incident in a point, the point of incidence is the unique (point) cycle incident with all three.

\textbf{Case 2} If the lines underlying the three spears form a triangle (that is, no two of the underlying lines are parallel), then four Euclidean circles will be tangent to all three lines - one inscribed in the triangle, and the other three outside of the triangle, one on each side. Depending
Figure 5: Cycles $c$ and $d$ touch in $A$ by being tangent with $A$ at the same point. No other spear can be incident with both cycles.

Figure 6: Some examples of three spears determining a unique cycle.
on the orientations of the spears, exactly one orientation of exactly one of these circles will touch all three spears. (The 8 possible cycles based on these four tangent circles correspond to the $2^4$ possible orientations of the set of three spears.)

**Case 3** If two of the lines underlying the three spears are parallel, there will be two Euclidean circles tangent to all three underlying lines. Depending again on the orientations of the spears, one orientation of one of these circles will be the unique cycle incident to all three spears.

2 *Touching.* Let $P$ and $Q$ be two spears, and $c$ be a cycle incident with $P$. We want to show there is a unique cycle $d$ through $Q$ that touches $c$ in $P$. Clearly, if $Q$ is in $c$, then $c = d$. Otherwise, suppose $Q$ is not incident with $c$. If $c$ is a point cycle (as in Figure 7), there will be two Euclidean circles tangent to the untouched spear through $c$. Depending on the orientations of the spears, one cycle overlaying one of those circles will be the unique cycle that touches all three. If $c$ is a proper cycle (i.e., an oriented circle), then treat its point of tangency with the spear it touches as a point cycle, as above; the unique cycle will be the one that touches the point cycle and both spears.

![Figure 7: Point $c$ is a point cycle touching $P$ but not $Q$. Depending on the orientations of $P$ and $Q$, one orientation of either $d'$ or $d''$ will touch all three of $c$, $P$, and $Q$.](image)

3 *Parallel Intersection.* We have to show that, given a spear $P$ and a cycle $c$, there is a unique spear on $c$ (tangent to it with agreeing orientation) parallel to $P$. If $P$ is on $c$, it is that unique spear. Suppose $P$ is not on $c$. In the Euclidean plane, a circle will have two tangent lines of any given
slope, so of the class of spears parallel to \( P \), two of them will be tangent to \( c \), and exactly one will have matching orientation.

4 Richness. This is another infinite model that easily satisfies a richness axiom.

We now turn to a brief discussion of the relationship between Laguerre planes and a couple of other areas of geometry.

6 Related Areas

6.1 Circle Planes

A Laguerre plane is a type of circle plane. A circle plane is an incidence structure of points and circles that contains up to two equivalence relations on the points, called parallelisms. As in a Laguerre plane, points are parallel to themselves; this is called the trivial parallelism. A plane that contains one parallelism beyond the trivial one is a Laguerre plane. (It was proved in Theorem 1 that parallelism is an equivalence relation in our axiom set.) A plane that has only the trivial parallelism is a Möbius plane, and a plane with two trivial parallelisms is a Minkowski plane. Circle planes obey the axioms given in section 2, plus the following axiom (which is irrelevant to Laguerre planes with their one parallelism):

Axiom 5. Two parallel classes with respect to different nontrivial parallelisms intersect in a unique point.

(For more on circle planes, see [9] or [7].)

6.2 The Generalized Quadrangle

The generalized quadrangle is an incidence structure - one that, unlike the projective plane, does not guarantee that any two points are collinear, but does guarantee that, if points \( P \) and \( Q \) are not collinear, and \( l \) is a line containing \( Q \), there is a unique point \( R \) and line \( m \) such that \( P \) is on \( m \), and \( R \) is on both \( l \) and \( m \). (This is true of opposing points of a quadrangle, hence “generalized quadrangle.” A projective plane is a “generalized triangle” along similar lines.)

The relationship between Laguerre planes and the Generalized Quadrangle is well-studied. Schroth [8] discusses the construction of a generalized quadrangle from a Laguerre plane; this can be done with a finite Laguerre Plane of odd order or with topological (roughly, “continuous”) Laguerre planes. Chapter 16 of Payne’s Topics in Finite Geometry [5] contains a more thorough discussion of the finite case.

There are numerous connections between Laguerre planes and other areas of geometry. For detailed discussions of Laguerre and other circle planes, see [5] or [9].
References


