1. Introduction

In this worksheet we will be exploring some proofs surrounding the Theorem of Desargues. This “theorem” plays an extremely important role in projective geometry, although it is not universally true. We will prove three propositions relating to the Theorem of Desargues in this worksheet:

1. The Theorem of Desargues implies its converse.
2. The Theorem of Pappus implies the Theorem of Desargues.
3. The Theorem of Desargues holds in any projective space of dimension 3 or greater.

In order to make the statement of the Theorem of Desargues easier, we make the following definition.

**Definition 1.** Two triangles $A, B, C$; and $A', B', C'$ are perspective from a point if $AA', BB',$ and $CC'$ are concurrent. They are perspective from a line if $AB \cap A'B', AC \cap A'C',$ and $BC \cap B'C'$ are collinear.

With these definitions, we have the following simple statement of the Theorem of Desargues.

**Theorem 2 (Theorem of Desargues).** If two triangles with distinct vertices $A, B, C$; and $A', B', C'$ are perspective from a point $V$ distinct from the vertices, then they are perspective from a line.

The theorem is represented as follows, where the shaded triangles are perspective from the point $V$ and from the line through $L, M,$ and $N$. If the Theorem of Desargues holds, $L, M,$ and $N$ will always be collinear; if the Theorem of Desargues does not hold, $L, M,$ and $N$ may not be collinear.

**Figure 1.** Desargues Configuration

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2. DESARGUES IMPLIES ITS CONVERSE

We assume that Desargues’ Theorem holds and show that its converse also holds.

(1) State the converse of Desargues’ Theorem.

(2) Draw two triangles in perspective from a line \( x \) by drawing \( x \) and then drawing lines \( a \) and \( a' \) meeting on \( x \), \( b \) and \( b' \) meeting on \( x \), and \( c \) and \( c' \) meeting on \( x \) so that the three points of intersection with \( x \) are distinct.

(3) Lightly shade the triangles enclosed by sides $a, b, c$ and $a', b', c'$.
(4) Draw lines $l = \langle a \cap b, a' \cap b' \rangle$, $m = \langle a \cap c, a' \cap c' \rangle$, and $n = \langle b \cap c, b' \cap c' \rangle$.
(5) If the converse of Desargues’ Theorem holds, what should be true of these lines?

They should be concurrent.

(6) There are two pairs of lines that we know meet on $n$. What are these pairs of lines?

$b, c$ and $b', c'$

(7) Are the triangles formed by $m, c, c'$ and $l, b, b'$ in perspective from a point?

Yes, from the point $a \cap a'$.

(8) From the Theorem of Desargues, what three points must now be collinear?

$b \cap c, b' \cap c'$, and $l \cap m$

(9) What does this mean for the lines $l, m,$ and $n$?

Since $b \cap c$ and $b' \cap c'$ are both on $n$, $l \cap m$ must also be on $n$ so these three lines must be concurrent.

(10) Conclude that the converse of Desargues’ Theorem also holds.

The triangles $abc$ and $a'b'c'$ that were in perspective from a line were necessarily in perspective from a point, so the converse of Desargues’ Theorem also holds.

A very similar proof to the above shows that the converse of Desargues’ Theorem implies Desargues’ Theorem.
3. **Pappus Implies Desargues**

We assume that Pappus’ Theorem holds and show that Desargues’ Theorem also holds.

(1) State Pappus’ Theorem.

Given lines \(l\) and \(l'\) with distinct points \(A, B, C\) on \(l\) and \(A', B', C'\) on \(l'\), all distinct from \(l \cap l'\), the points \(L = AB' \cap A'B, M = AC' \cap A'C,\) and \(N = BC' \cap B'C\) are collinear.

(2) Draw triangles \(A, B, C\) and \(A', B', C'\) with distinct vertices in perspective from a point \(V\) distinct from all of the vertices.

(3) Define the points \(S = B'C' \cap AC, L = BC \cap B'C', T = B'A \cap CC',\) and \(U = BA \cap VS\), drawing any necessary additional lines.

(4) Use Pappus’ Theorem to show that \(U, T,\) and \(L\) are collinear.

Note that the points \(V, B, B'\) are collinear; the points \(A, S, C\) are collinear; and all are distinct from the intersection of their lines. Thus, Pappus’ Theorem implies that \(U = AB \cap VS, T = AB' \cap VC,\) and \(L = BC \cap B'S\) are collinear.
(5) Make a fresh drawing showing the points $A$, $B$, $C$, $A'$, $B'$, $C'$, $V$, $U$, $T$, $L$, and $S$.

(6) Define the points $P = B'A' \cap VS$ and $M = CA \cap C'A'$.

(7) Use Pappus’ Theorem to show that $P$, $M$, and $T$ are collinear.

Note that the points $V$, $A$, $A'$ are collinear; the points $B'$, $C'$, $S$ are collinear; and all are distinct from the intersection of their lines. Thus, Pappus’ Theorem implies that $T = VC' \cap AB'$, $P = VS \cap B'A'$, and $M = AS \cap A'C'$ are collinear.
(8) Make another fresh drawing showing the points $A, B, A', B', T, U, P,$ and $S$ (drawing other points as needed to properly locate these points).

(9) Define the points $N = AB \cap A'B'$.

(10) Use Pappus’ Theorem to show that $L, M, \text{and } N$ are collinear.

Note that the points $U, S, P$ are collinear; the points $B', T, A$ are collinear; and all are distinct from the intersection of their lines. Thus, Pappus’ Theorem implies that $L = UT \cap B'S$, $N = UA \cap B'P$, and $M = SA \cap TP$ are collinear.

(11) Conclude that the triangles $A, B, C$ and $A', B', C'$ are perspective from the line $LM$.

We showed that $L, M, \text{and } N$ were collinear, and defined the points $L = BC \cap B'C'$, $M = AC \cap A'C'$, and $N = AB \cap A'B'$. Thus, the definition of perspective from a line holds for the triangles and the line $LM$.

(12) Conclude that Desargues’ Theorem holds.

Since we started with the assumption that $A, B, C$ and $A', B', C'$ were perspective from a point and showed them to be perspective from a line, Desargues’ Theorem holds.
4. Higher-Dimensional Spaces are Desarguesian

We now assume that our lines and points are in a projective space of dimension at least three.

4.1. Non-coplanar Triangles. We first show that Desargues’ Theorem holds for two triangles in perspective from a point that are not contained in a common plane.

1. Let triangles $A, B, C$ and $A', B', C'$ with distinct vertices be in distinct planes $\pi$ and $\alpha$ respectively and be perspective from $V$.

2. Show that $\dim (\pi \oplus \alpha) = 3$.

Note that since $\pi \neq \alpha$, $\dim (\pi \oplus \alpha) \geq 3$. Now consider $\pi \oplus V$. Since $V \notin \pi$, $\dim (\pi \oplus V) = 3$. Define $\Sigma = \pi \oplus V$, and consider $\Sigma \cap \alpha$. Note that the lines $VA = VA'$, $VB = VB'$, and $VC = VC'$ are all contained in $\Sigma$. Then so must be the line $A'B'$, a line which is also contained in $\alpha$, but does not contain the point $C'$ of $\Sigma \cap \alpha$. So $\dim (\Sigma \cap \alpha) = 2$, and therefore $\alpha \subseteq \Sigma$. So $\Sigma$ is a three dimensional space containing both $\pi$ and $\alpha$, and therefore $\pi \oplus \alpha$ has dimension three.

3. Deduce that $\pi \cap \alpha$ is a line.

$\dim (\pi \cap \alpha) = 2 + 2 - \dim (\pi \oplus \alpha) = 2 + 2 - 3 = 1$, so $\pi \cap \alpha$ is a line.

4. Show that $\dim (AB \cap A'B') = \dim (AC \cap A'C') = \dim (BC \cap B'C') = 0$. That is, show that $AB$ meets $A'B'$ in a point, $AC$ meets $A'C'$ in a point, and $BC$ meets $B'C'$ in a point.

It is easiest to apply the second axiom of projective space: Since $AA'$ intersects $BB'$ in the point $V$, $AB$ intersects $A'B'$ in a point. Similarly, $AC$ intersects $A'C'$ in a point and $BC$ intersects $B'C'$ in a point.

5. Argue that $AB \cap A'B'$, $AC \cap A'C'$, and $BC \cap B'C'$ are all in $\pi \cap \alpha$.

Since $AB \subseteq \pi$ and $A'B' \subseteq \alpha$, $AB \cap A'B' \subseteq \pi$ and $AB \cap A'B' \subseteq \alpha$, so that $AB \cap A'B' \subseteq \pi \cap \alpha$. Similarly, $AC \cap A'C' \subseteq \pi \cap \alpha$ and $BC \cap B'C' \subseteq \pi \cap \alpha$.

6. Conclude that the Theorem of Desargues holds for non-coplanar triangles.

The points $AB \cap A'B'$, $AC \cap A'C'$ and $BC \cap B'C'$ are collinear in the line $\pi \cap \alpha$. 
4.2. Coplanar Triangles. We use the fact that Desargues’ Theorem holds for non-coplanar triangles to show that Desargues’ Theorem holds for coplanar triangles in a projective space of dimension at least three.

1. Let $A$, $B$, $C$ and $A'$, $B'$, $C'$ be triangles with distinct vertices in a plane $\pi$ and in perspective from the point $X$ distinct from the vertices. Let $l$ be a line through $X$ not in $\pi$.
2. Show that $l$ intersects $\pi$ only in the point $X$ (it is not a line of $\pi$), all other points of $l$ are not in $\pi$.

Since $l$ intersects $\pi$ only in the point $X$ (it is not a line of $\pi$), all other points of $l$ are not in $\pi$.

One of the projective space axioms states that every line has at least three points, so there exist at least two points $O$, $O'$ of $l$ not in $\pi$.

3. Show that $OA'$ intersects $O'A$ in a point $A_1$, $OB'$ intersects $O'B$ in a point $B_1$, and $OC'$ intersects $O'C$ in a point $C_1$.

The lines $l$ and $AA'$ intersect in a point and therefore span a plane in which both $OA'$ and $O'A$ lie. Thus, $OA'$ intersects $O'A$ in a point $A_1$. Similarly, $OB'$ intersects $O'B$ in a point $B_1$ and $OC'$ intersects $O'C$ in a point $C_1$.

4. Show that $A_1$, $B_1$, $C_1$ is a triangle not in $\pi$ and call the plane containing this triangle $\alpha$.

The line $OA'$ intersects $\pi$ only in the point $A'$, and the line $O'A$ intersects $\pi$ only in the point $A$, distinct from $A'$. So the intersection point $A_1$ of $OA'$ and $O'A$ certainly cannot be a point of $\pi$. Thus, the triangle $A_1$, $B_1$, $C_1$ is not in $\pi$.

5. Show that the points $A_1$, $B_1$, $C_1$, $A'$, $B'$, $C'$, $A$, $B$, and $C$ are all distinct.

By assumption, $A$, $B$, $C$, $A'$, $B'$, $C'$ are all distinct. The argument in the last question shows that $A_1$ is not in the plane $\pi$ and holds equally well for $B_1$ and $C_1$, so these points are distinct from $A$, $B$, $C$, $A'$, $B'$, $C'$. Further, if $A_1 = B_1$, $OA' = OB'$ and $A' = B'$, so $A_1 \neq B_1$, and in the same way $A_1 \neq C_1$ and $B_1 \neq C_1$. So all points are distinct.

6. Use Desargues’ Theorem for non-coplanar triangles to show that $A_1B_1 \cap AB$, $A_1C_1 \cap AC$, and $B_1C_1 \cap BC$ are collinear in $\pi \cap \alpha$.

The triangles $A_1$, $B_1$, $C_1$ and $A$, $B$, $C$ are non-coplanar and are in perspective from the point $O'$. Thus, they are in perspective from the line $\pi \cap \alpha$, and $A_1B_1 \cap AB$, $A_1C_1 \cap AC$, and $B_1C_1 \cap BC$ are collinear in the line $\pi \cap \alpha$.

7. Use Desargues’ Theorem for non-coplanar triangles to show that $A_1B_1 \cap A'B'$, $A_1C_1 \cap A'C'$, and $B_1C_1 \cap B'C'$ are collinear in $\pi \cap \alpha$.

The triangles $A_1$, $B_1$, $C_1$ and $A'$, $B'$, $C'$ are non-coplanar and are in perspective from the point $O$. Thus, they are in perspective from the line $\pi \cap \alpha$, and $A_1B_1 \cap A'B'$, $A_1C_1 \cap A'C'$, and $B_1C_1 \cap B'C'$ are collinear in the line $\pi \cap \alpha$.

8. Argue that then $AB \cap A'B'$, $AC \cap A'C'$, and $BC \cap B'C'$ are collinear in $\pi \cap \alpha$.

Since none of $A_1$, $B_1$, $C_1$ lie in $\pi$, the lines $A_1B_1$, $A_1C_1$, and $B_1C_1$ each intersect $\pi \cap \alpha$ in a unique point. Thus, $A_1B_1$ intersects $AB$ and $A'B'$ in the same point $AB \cap A'B'$ in $\pi \cap \alpha$. Similarly, $A'C \cap A'C' \in \pi \cap \alpha$ and $BC \cap B'C' \in \pi \cap \alpha$.

9. Conclude that Desargues’ Theorem holds for coplanar triangles in a projective space of dimension at least three.
Starting with coplanar triangles perspective from the point $X$, we showed that the existence of the third dimension forced the triangles to be perspective from the line $\pi \cap \alpha$.

(10) Conclude that Desargues’ Theorem holds universally in projective spaces of dimension at least three.

Since Desargues’ Theorem holds universally for non-coplanar triangles and for coplanar triangles in projective spaces of dimension at least three, it holds universally in projective spaces of dimension at least three.