Nadia Hamoudi

The Chang graphs

Abstract

Chang showed that there are up to isomorphism four strongly regular graphs with parameters $v = 28$, $k = 12$, $\lambda = 6$, $\mu = 4$, namely T(8) and three other graphs, known as the *Chang graphs*. He proved that the these three graphs can be obtained by Seidel switching from T(8) (the line graph of K_8), with respect to the edge set of 4 K_2, $K_3 + K_5$ and C_8.

Introduction

The work of Shrikhande (1959), A.J.Hoffman in the late 1950's and Chang (1959, 1960) have contributed to the development of Seidel's proof of the classification of strongly regular graphs with least eigenvalues -2. Seidel proved that strongly regular graphs with least eigenvalues -2 are: the triangular graphs, the lattice graphs, the cocktail party graphs, the Petersen graph, the complement of the Clebsch graph, the Shrikhande graph, the complement of the Schläfli graph, or the three Chang graphs.

Switching a graph with respect to a set Y of vertices replaces all edges between Y and its complement with non-edges, and leaves edges within Y and outside Y unchanged. The set of all graphs on a vertex set X falls into equivalence classes of size 2^{n-1}, where two graphs are equivalent if one of them can be obtained from the other by switching with respect to some subset Y.

The triangular graph T(m) where $m \geq 4$ is the graph with vertex set the 2-element subsets of the set {1, 2, …, m} where two vertices are adjacent if and only if they are not disjoint. T(m) is strongly regular with the parameters ($\frac{1}{2}$ m(m-1), 2(m-2), m-2, 4).

Chang proved that T(m) are uniquely determined by their parameters when m is not 8. For m = 8 there are four pairwise non-isomorphic strongly regular graphs with parameters (28, 12, 6, 4), T(8) and three other graphs that have the same parameters as T(8), named after Chang. The Chang graphs can be obtained from T(m) by Seidel switching .

The switching sets

Since T(8) is the set of 2-subsets of {1, 2, …, 8} and adjacency between two vertices is equivalent to the two vertices sharing a point, then a subset of the vertex set of T(8) can be seen as the edge set of a regular graph G with 8 vertices. Switching with respect to a set Y, and with respect to its complement is the same operation since the edges in Y and in the complement will be left unchanged, and the edges between Y and its complement are the same as the edges between the complement and Y. Therefore when studying the switching sets we may assume that Y is smaller than its complement which means that the graph G has at most 14 vertices and thus has degree at most 3.

If Y is a switching set of T(8) then for a vertex y in Y of degree d in Y, y is adjacent to 12-d vertices in the complement of Y, $\overline{Y}$ and not adjacent to (28 - |Y|) - (12 – d) vertices in $\overline{Y}$. This implies that the switching will result in a graph in which y is adjacent to d+[(28 - |Y| - (12 – d)) – (12 – d)] vertices, which means that the degree of a vertex in the resulting graph depends only on the choice of Y and not on the vertex itself. The number of neighbors in Ý of a vertex in Y is a constant independent of the choice of the vertex. In this case we say that {Y, $\overline{Y}$ } is a equitable partition of T(8).

The fact that the partition of (T(8) = L(K_8)) into Y and $\overline{Y}$ is equitable implies that the corresponding partition of the edges of K_8 into two sets X_1 and X_2 satisfies that L(X_1) and L(X_2) are regular.

Theorem: The line graph of a connected graph is regular if the graph is regular or it's bipartite semiregular.

Definition: A graph is bipartite semiregular if it has a proper 2-coloring such that the vertices of one color have the same degree.

If one of X_1 or X_2 is bipartite semiregular than the other one would have to be the union of two cliques of different order, this means the line graph of it is not regular which contradicts our assumption. So both X_1 and X_2 are regular graphs. If X_1 is an r-regular graph then |Y| = r (8/2) = 4r.

The preceding discussion help us state the possible switching sets Y which switch T(8) into a regular graph of degree 12, as follows:

1. If r = 1.
   The edge set of the disjoint union of four K_2 graphs, in this case we get the graph Chang 2.

2. If r = 2.
   The edge set is either
   C_8 which results in the graph Chang 2.
   $C_5 \cup C_3$ which gives Chang 3.
   $C_4 \cup C_4$ which gives Chang 1.

3- If r = 3.
switching T(8) with respect to the edge set of 2 K_4 gives T(8).
Switching T(8) with respect to other 3-regular graphs give graphs isomorphic to already obtained ones.

T(8) and the three Chang graphs are pairwise nonisomorphic.