THE HIGMAN-SIMS GRAPH

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Abstract. We review known constructions of the Higman-Sims strongly regular graph, as well as significant properties of this graph, including its relations to other important graphs.

1. Introduction

Donald G. Higman and Charles C. Sims first introduced the Higman-Sims Graph in 1968 [8] as part of the construction of a simple group of order $44,352,000$ (the Higman-Sims simple group). The graph is defined in the following manner.

Definition 1.1. Let $\mathscr{D}$ be the unique $3-(22,6,1)$ design with point set $\mathscr{P}$ and block set $\mathscr{B}$. The Higman-Sims graph Γ is the graph with vertex set $V = \mathscr{P} \cup \mathscr{B} \cup \{\infty\}$ and edge set $E = E_1 \cup E_2 \cup E_3$, where E_1, E_2, and E_3 are the following:

(1) $E_1 = \{(\infty, P) \,|\, P \in \mathscr{P}\}$.
(2) $E_2 = \{(P, b) \,|\, P \in \mathscr{P} \quad b \in \mathscr{B} \quad P \in b\}$.
(3) $E_3 = \{(b, b') \,|\, \{b, b'\} \subseteq \mathscr{B} \quad b \cap b' = \emptyset\}$.

Theorem 1.2. *The Higman-Sims graph is a strongly regular graph with parameters* $(100, 22, 0, 6)$

Proof. The design $\mathscr{D}$ has $|\mathscr{P}| = 21$ and $|\mathscr{B}| = 77$, so that $|V| = 21 + 77 + 1 = 100$.

Every point in $\mathscr{P}$ lies in precisely 21 blocks, so that the corresponding vertices are each incident with 22 other vertices when ∞ is included. Every block in $\mathscr{B}$ contains precisely 6 points and is disjoint from precisely 16 blocks, so that the corresponding vertices are each incident with 22 other vertices. Finally, ∞ is incident with precisely the 22 points of $\mathscr{P}$, so that Γ is regular of degree k.

Since ∞ is only adjacent to points and no two points are adjacent, ∞ has no common neighbors with any adjacent point. Given a point P and block b such that

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$P \in b$, b is not adjacent to ∞, nor disjoint from any other block containing P. Finally, there is no triple of mutually disjoint blocks in $\mathscr{D}$, so that Γ is triangle free.

Any two points lie in 5 common blocks and also share ∞ as a common neighbor. Any two intersecting blocks share exactly 2 points and are disjoint from 4 common blocks. Any block contains 6 points, all adjacent to ∞, so that $\mu = 6$. □

Theorem 1.3. [5] *The strongly-regular graph with parameters* $(100, 22, 0, 6)$ *is unique.* □

2. Constructions

The original construction of Higman and Sims, based on the unique $3-(22, 6, 1)$ design can be modified in the following manner (due to Biggs and White [1]) to incorporate the construction of the $3-(22, 6, 1)$ design from $PG(2, 4)$.

Theorem 2.1. *Let* $\mathscr{P}$ *be the set of points and* $\mathscr{L}$ *the set of lines of* $PG(2, 4)$*, and let* $\mathscr{H}$ *be an equivalence class of hyperovals meeting in an even number of points. Let* Γ *be the graph with vertex set* $\mathscr{P} \cup \mathscr{L} \cup \mathscr{H} \cup \{\infty, \infty'\}$ *and with edge set* $E_1 \cup E_2 \cup E_3 \cup E_4 \cup E_5 \cup E_6 \cup E_7$*, with* E_i *defined as follows:*

(1) $E_1 = \{(\infty, \infty')\}$.
(2) $E_2 = \{(\infty, P) \,|P \in \mathscr{P}\}$.
(3) $E_3 = \{(\infty', l) \,|l \in \mathscr{L}\}$.
(4) $E_4 = \{(P, l) \,|P \in \mathscr{P} \quad l \in \mathscr{L} \quad P \in l\}$.
(5) $E_5 = \{(p, H) \,|P \in \mathscr{P} \quad H \in \mathscr{H} \quad P \in H\}$.
(6) $E_6 = \{(l, H) \,|l \in \mathscr{L} \quad H \in \mathscr{H} \quad l \cap H = \emptyset\}$.
(7) $E_7 = \{(H, H') \,|H \in \mathscr{H} \quad H' \in \mathscr{H} \quad H \cap K = \emptyset\}$.

Then Γ *is the Higman-Sims graph.* □

Following Hughes [9], the Higman-Sims graph can also be constructed from a polarity of a semi-symmetric 3-design.

Theorem 2.2. *There exists a unique semi-symmetric 3-design for* $\lambda = 2$ *with 100 points and 22 points on each block. This semi-symmetric 3-design admits a polarity* α*.*

Let Γ *be the graph where each vertex is formed by identifying a point and its image under* α *and where adjacency is determined by incidence in the design.*

Then Γ *is the Higman-Sims graph.* □

Work of Mathon and Street [12] uses two-fold triple systems and Steiner triple systems on seven points to construct the Higman-Sims graph.

Definition 2.3. Let $\mathscr{P}$ be the set whose elements are the 35 triples of elements in $\mathbb{Z}_7$ and the 15 distinct Steiner triple systems on $\mathbb{Z}_7$ in a given orbit under A_7. Let each $\mathscr{B}_i$ be a collection of 15 element subsets of P defined as follows:

(1) B_1 consists of 42 blocks, each of which contains the 10 triples from a two-fold triple system that shares no triple with five of the Steiner triple systems together with those five Steiner triple systems.
(2) B_2 consists of 7 blocks, each of which contains the 15 triples containing a fixed element of $\mathbb{Z}_7$.
(3) B_3 consists of one block, containing the 15 Steiner triple systems.
(4) B_4 consists of 15 blocks, each of which contains the 7 triples of a particular Steiner triple system, together with the 8 Steiner triple systems containing none of these triples.
(5) B_5 consists of 35 blocks, each of which contains the 12 triples intersecting a given triple in exactly two elements of $\mathbb{Z}_7$, as well as the three Steiner triple systems containing the given triple.

The design $\mathscr{D}_1$ has as points $\mathscr{P}$ and as blocks $\mathscr{B}_1 \cup \mathscr{B}_2 \cup \mathscr{B}_3$, and the design $\mathscr{D}_2$ has as points $\mathscr{P}$ and as blocks $\mathscr{B}_4 \cup \mathscr{B}_5$.

Theorem 2.4. *Let* Γ *be the graph with vertex set* $\mathscr{B}_1 \cup \mathscr{B}_2 \cup \mathscr{B}_3 \cup \mathscr{B}_4 \cup \mathscr{B}_5$ *and with edge set* $E_1 \cup E_2 \cup E_3$*, with* E_i *defined as follows:*

(1) $E_1 = \{(B, B') \,|B \in \mathscr{B}_1 \cup \mathscr{B}_2 \cup \mathscr{B}_3, \quad B' \in \mathscr{B}_1 \cup \mathscr{B}_2 \cup \mathscr{B}_3, \quad B \cap B' = \emptyset\}$
(2) $E_2 = \{(B, B') \,|B \in \mathscr{B}_4 \cup \mathscr{B}_5, \quad B' \in \mathscr{B}_4 \cup \mathscr{B}_5, \quad B \cap B' = \emptyset\}$
(3) $E_3 = \{(B, B') \,|B \in \mathscr{B}_1 \cup \mathscr{B}_2 \cup \mathscr{B}_3, \quad B' \in \mathscr{B}_4 \cup \mathscr{B}_5, \quad |B \cap B'| = 8\}$

Then Γ *is the Higman-Sims graph.* □

Mimicking a construction of the Hoffman-Singleton graph, Hafner [6] gives a construction parametrizing the vertices by the group $\mathbb{Z}_4 \times \mathbb{Z}_5 \times \mathbb{Z}_5$ with adjacencies determined by linear and quadratic equations.

Theorem 2.5. *Let* Γ *be the graph with vertex set the elements of the group* $\mathbb{Z}_4 \times \mathbb{Z}_5 \times \mathbb{Z}_5$ *and edge set* $\bigcup_{i=1}^{10} E_i$*, with* E_i *defined in the following way:*

(1) $E_1 = \{((0,x,y),(0,x,y'))\,|y-y' = \pm 1\}$.
(2) $E_2 = \{((1,m,c),(1,m,c'))\,|c-c' = \pm 2\}$.
(3) $E_3 = \{((2,A,B),(2,A,B'))\,|B-B' = \pm 1\}$.
(4) $E_4 = \{((3,a,b),(3,a,b'))\,|b-b' = \pm 2\}$.
(5) $E_5 = \{((0,x,y),(1,m,c))\,|y = mx + c\}$.
(6) $E_6 = \left\{((1,m,c),(2,A,B))\,|c = 2(m-A)^2 + B\right\}$.
(7) $E_7 = \left\{((2,A,B),(3,a,b))\,|B = 2A^2 + 3aA - a^2 + b\right\}$.
(8) $E_8 = \left\{((3,a,b),(0,x,y))\,|y = (x-a)^2 + b\right\}$.
(9) $E_9 = \left\{((0,x,y),(2,A,B))\,|y = 3x^2 + Ax + B \pm 1\right\}$.
(10) $E_{10} = \left\{((1,m,c),(3,a,b))\,|c = m^2 - am + b \pm 2\right\}$.

Then Γ *is the Higman-Sims graph.* □

The Higman-Sims graph can also be realized from partial difference sets in non-Abelian groups of order 100. Jørgensen and Klin [10], Heinze [7], and Praeger and Schneider (private communication cited in [6]) independently discovered this realization of the graph.

Definition 2.6. Given a group G and a set $S \subset G$ such that $S^{(-1)} = S$ and $1 \notin S$, the Cayley graph Γ is the graph with vertex set H and edge set $\left\{(x,y)\,|x^{-1}y \in S\right\}$.

A Cayley graph is strongly regular if and only if S is a regular partial difference set in G. In this case, Γ has the same parameters as S. Certain partial difference sets in certain groups of order 100 yield the Higman-Sims graph.

A further construction of the Higman-Sims graph, due independently to Wilson [14] and Yoshiara [15] uses structures in the Leech lattice.

3. Related Graphs

The Higman-Sims graph is closely related to several other graphs. The M_{22} graph is the unique strongly regular graph with parameters $(77, 16, 0, 4)$ [3].

Theorem 3.1. *The* M_{22} *graph is the induced subgraph of the Higman-Sims graph on the vertices that are neither adjacent nor equal to a given vertex.* □

Brouwer and Haemers [4] embed the Gewirtz graph into the Higman-Sims graph in a similar manner.

Theorem 3.2. *The Gewirtz graph is the induced subgraph of the Higman-Sims graph on the vertices that are adjacent to neither end-vertex of an edge.* □

Brouwer [2] notes the following property for the vertices adjacent to a given edge.

Theorem 3.3. *The induced subgraph of the Higman-Sims graph on the vertices adjacent to an end-vertex of a given edge is the point-line incidence graph of* $PG(2,4)$.

Perhaps the most significant subgraph of the Higman-Sims graph is the Hoffman-Singleton graph. In fact, Robertson's construction [13] of the Hoffman-Singleton graph is plainly visible as a subset of Hafner's construction [6] of the Higman-Sims graph. However, the relation is stronger. The following properties are noted by Hafner [6].

Theorem 3.4. *The Higman-Sims graph contains 704 subgraphs isomorphic to the Hoffman-Singleton graph. These fall into 352 disjoint pairs (so that the induced subgraph on the vertices not in a given Hoffman-Singleton subgraph is another Hoffman-Singleton subgraph), which fall into two equivalence classes under the action of the Higman-Sims simple group on the Higman-Sims graph.* □

These are the most immediate and interesting subgraphs of the Higman-Sims graph.

4. Automorphisms

As indicated by the initial construction of the Higman-Sims graph, the automorphism group is of significant interest. Higman and Sims [8] constructed the graph as a means of constructing the unique simple group of order $44,352,000$.

The determination of the size of the automorphism group relies on two distinct properties:

(1) The stabilizer of a vertex is isomorphic to the automorphism group of the $3-(22,6,1)$ design, a degree two extension of the Mathieu group M_{22}, and therefore has order $887,040$.
(2) The automorphism group is transitive on vertices.

From these two properties it is clear that the full automorphism group of the graph has order $88,704,000$. However, as noted in [8], this group must contain an odd permutation. Hence, it contains a subgroup of index two, which is the Higman-Sims simple group of order $44,352,000$.

Magliveras [11] describes the maximal subgroups and orbit structures of the Higman-Sims group and Brouwer [2] describes the corresponding actions on the graph as the following.

The stabilizer of any vertex has two additional orbits of orders 22 and 77. The orbit of order 77 induces the M_{22} graph as previously mentioned. The orbit of order 22 is the open neighborhood of the vertex.

The group is transitive on edges of the graph, with the stabilizer of an edge having two additional orbits of orders 42 and 56. The induced subgraphs are, respectively, the point-line incidence graph of $PG\,(2,4)$ and the Gewirtz graph.

The group is transitive on pairs of non-adjacent vertices in the graph, with the stabilizer of such a pair having three additional orbits of orders 6, 32, and 60. The induced subgraph on the orbit of order 32 is the folded 6-cube, while that on the orbit of order 60 is the induced subgraph on the non-neighbors of a given vertex in the $M_2 2$ graph.

The remaining maximal subgroups stabilize the following structures:

(1) A split into two Hoffman-Singleton graphs.
(2) The complement of the point plane incidence graph of $PG\,(3,2)$.
(3) A 2-coclique extension of the Petersen graph.
(4) A partition into four $5C_5$.

as well as some yet more obscure structures.

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