Synthetic Geometry

1.6 Affine Geometries
Affine Geometries

Def: Let \( P \) be a projective space of dimension \( d > 1 \) and \( H_\infty \) a hyperplane. Define \( A = P \setminus H_\infty \) as follows:

1. The **points** of \( A \) are the points of \( P \setminus H_\infty \).
2. The **lines** of \( A \) are the lines of \( P \) not in \( H_\infty \).
3. The **t-dimensional subspaces** of \( A \) are the t-dimensional subspaces of \( P \) which are not contained in \( H_\infty \).
4. Incidence is inherited.

The rank 2 geometry of points and lines of \( A \) is called an **affine space** of dimension \( d \). An **affine plane** is an affine space of dimension 2. The set of all subspaces of \( A \) is called an **affine geometry**.
Affine Geometries

For fixed $t$ ($1 \leq t \leq d-1$), the rank 2 geometry consisting of the points and $t$-dimensional subspaces of $A$ is denoted by $A_t$ (thus affine space is the geometry $A_1$). Subspaces of an affine space are called flats.

$H_\infty$ is called the hyperplane at infinity and its points are called points at infinity (sometimes improper points).

$P$ is called the projective closure of $A$.  

Examples of Affine Geometries

Examples:
1. Fano Plane

 Affine Plane of order 2

2. Affine Plane of order 3
Examples of Affine Geometries

3. Consider the example of section 1.4. Let $\mathbf{H}_\infty$ be the plane $0001^\top$. The points of this plane (points at infinity) are those with last coordinate 0. The points of the affine space $\mathbf{P}\setminus\mathbf{H}_\infty$ are thus: $0001, 0011, 0101, 0111, 1001, 1011, 1101$, and $1111$. The projective plane $1000^\top$ becomes the affine plane whose points are: $0001, 0011, 0101$ and $0111$. The line labeled a of the projective space becomes the affine line whose points are $0001$ and $1001$. 
Parallelism

Def: Let \( G = (\mathcal{P}, \mathcal{B}, I) \) be a rank 2 geometry. A parallelism of \( G \) is an equivalence relation \( \parallel \) on the block set \( \mathcal{B} \) satisfying Playfair's axiom (i.e., given a point \( P \) and a block \( B \) not containing \( P \), there is a unique block \( B' \) containing \( P \) with \( B' \parallel B \)).

Theorem 1.6.1: For \( t \in \{1, \ldots, d-1\} \), \( A_t \) has a parallelism.

Pf: Let \( \mathcal{P} \) be the projective space belonging to \( A \) with \( \mathcal{H}_\infty \) the hyperplane at infinity. Consider two \( t \)-dimensional subspaces \( U \) and \( W \) of \( A \). By definition, these are \( t \)-dimensional subspaces of \( \mathcal{P} \) not contained in \( \mathcal{H}_\infty \). Now \( U \cap \mathcal{H}_\infty \) and \( W \cap \mathcal{H}_\infty \) are subspaces of dimension \( t-1 \) in \( \mathcal{H}_\infty \) (1.3.12)
Parallelism

**Theorem 1.6.1:** For \( t \in \{1, \ldots, d-1\} \), \( A_t \) has a parallelism.

**Pf (cont.):** We define
\[
U \parallel W \iff U \cap \mathcal{H}_\infty = W \cap \mathcal{H}_\infty
\]
and shall show that \( \parallel \) is a parallelism. It is clear that \( \parallel \) is an equivalence relation, being defined by an equality.

We verify Playfair's axiom. Let \( U \) be a \( t \)-dimensional subspace of \( A \) and \( P \) a point of \( A \) not contained in \( U \). Let \( V = U \cap \mathcal{H}_\infty \) and note that it is a \((t-1)\)-dimensional subspace of \( \mathcal{H}_\infty \).

We see that any \( t \)-dimensional subspace \( W \) through \( P \) that is parallel to \( U \) must contain \( V \) and \( P \). Since \( <V,P> \) is a \( t \)-dimensional subspace of \( P \) which must be contained in \( W \), we have \( W = <V,P>_P \), by 1.3.9, so \( W \) exists and is unique. \( \square \)
Natural Parallelism

The parallelism of the above theorem is called the *natural parallelism*.

For the natural parallelism, any two distinct parallel t-flats span a (t+1)-flat. Two subspaces of arbitrary dimension are parallel if one is parallel to a subspace of the other. Thus, if $H_\infty$ is the hyperplane at infinity of $P$, then subspaces $U$ and $W$ are parallel if $U \cap H_\infty \subseteq W \cap H_\infty$, or vice-versa. In particular, a line $g$ is parallel to a hyperplane $H$ if $g$ is parallel to some line $h$ of $H$, meaning that $g \cap h \in H_\infty$. 
Lemma 1.6.2

Lemma 1.6.2: Let $A = P \setminus H_\infty$ where $P$ is a $d$-dimensional projective space.

   a. Each line not parallel to a hyperplane $H$ meets $H$ in precisely one point of $A$.
   b. If $d = 2$, any two non-parallel lines intersect in a point of $A$.

Pf: (a) Let $g$ be a line and $H$ a hyperplane of $A$ that are not parallel. Then $g$ intersects $H_\infty$ in a point outside $H \cap H_\infty$. By 1.3.12, $g$ and $H$ intersect at some point of $P$, which must therefore be in $A$.

   (b) When $d = 2$, a hyperplane is a line, so this follows immediately from (a).

$\square$
Properties of an Affine Plane

**Corollary 1.6.3**: Any affine plane \( A \) satisfies:
1. Through any two distinct points there passes exactly one line.
2. Playfair's axiom. Through a point \( P \) there is a unique line with no point in common with a given line \( g \) not incident with \( P \).
3. There exist 3 points not on a common line.

**Pf:** Let \( A = \mathbf{P} \setminus g_\infty \) where \( \mathbf{P} \) is a projective plane and \( g_\infty \) is the line at infinity of \( A \).

1. By axiom 1, any two points of \( \mathbf{P} \) lie on precisely one line. Points of \( A \) are points of \( \mathbf{P} \).
2. (2) follows from Theorem 1.6.1 and Lemma 1.6.2.
3. (3) By axiom 3, \( g_\infty \) has at least 3 points. Through any point \( P \) of \( A \) there are at least two lines, with each having a point \( P_i \) in \( A \) other than \( P \). So, \( P, P_1 \) and \( P_2 \) are three noncollinear points of \( A \). \( \Box \)
Theorem 1.6.4: Let $S = (P, L, I)$ be a rank 2 geometry satisfying 1), 2) and 3) above. Then $S$ is an affine plane.

Pf: We have to show that there is a projective plane $P$ containing a line $g_\infty$ such that $S = P \setminus g_\infty$. This will require “adding” points to $S$.

Define a relation $\parallel$ on the set of lines of $S$ by

$$g \parallel h \iff g = h \text{ or } (g) \cap (h) = \emptyset.$$

We first show that $\parallel$ is a parallelism.

That $\parallel$ is a reflexive and symmetric relation follows from the definition. Suppose $g$, $h$ and $k$ are lines with $g \parallel h$, and $h \parallel k$. If $g = k$ or $(g)$ and $(k)$ are disjoint, then they are parallel, so suppose that they are distinct and have a point $P$ in common. We can also assume that $g \neq h$, for otherwise $h \parallel k$ implies $g \parallel k$. Since $P I g$, $P$ is not on $h$, and 2) applied to $P$ and $h$ gives $g = k \rightarrow \leftarrow$. Thus $g \parallel k$. 

Theorem 1.6.4: Let $S = (P, L, I)$ be a rank 2 geometry satisfying 1), 2) and 3) above. Then $S$ is an affine plane.

Proof (cont.): Thus, $||$ is an equivalence relation and it follows from 2) that $||$ is a parallelism. We have also shown that any two nonparallel lines intersect each other in (precisely) one point.

The equivalence classes of this equivalence relation are called parallel classes. By Playfair's axiom, each point of $S$ is on precisely one line of each parallel class.

We are now going to define a new geometry $P$ where:

- points of $P$ = points of $S$ and the parallel classes of $S$.
- lines of $P$ = lines of $S$ and one new line $g_\infty$.

Incidence of $P$ is $I^*$ where:

$P I^* l$ if $P I l$ in $S$ or $l \in P$ if $P$ is a parallel class, for $l \neq g_\infty$ and $P I^* g_\infty$ iff $P$ is a parallel class.
Affine Planes

Theorem 1.6.4: Let $S = (P, L, I)$ be a rank 2 geometry satisfying 1), 2) and 3) above. Then $S$ is an affine plane.

\textit{Pf (cont.):} We now need to show that $P$ is a projective plane.

\textit{Axiom 1:} Any two points of $S$ are joined by a line of $S$, and no other line since $g_\infty$ has no points of $S$. Two parallel classes of $S$ are only on the line $g_\infty$. A point $P$ of $S$ and a parallel class $\Pi$ of $S$ are both on the unique line of $\Pi$ which contains $P$.

\textit{Axiom 2':} $g_\infty$ intersects any line $g$ of $S$ in the parallel class which contains $g$. Any two lines of $S$ are either parallel, in which case they meet at the parallel class which contains them both, or they meet at a point of $S$.

\textit{Axiom 3:} By 3) there exist 3 noncollinear points, which determine 3 lines, no two of which are parallel. Thus, there must be at least 3 parallel classes, so $g_\infty$ has at least 3 points.
Affine Planes

**Theorem 1.6.4:** Let $S = (\mathcal{P}, \mathcal{L}, I)$ be a rank 2 geometry satisfying 1), 2) and 3) above. Then $S$ is an affine plane.

**Pf (cont.):** Any other line is a line of $S$ which lies in a unique parallel class and so has a point of $g_\infty$ on it in $P$. To prove the axiom we need only show that each line of $S$ contains 2 points of $S$. Let $P$, $Q$ and $R$ be the three given noncollinear points. Any line $g$ of $S$ as a line of $P$ which does not pass through any of these points, must meet each of the 3 lines they determine at distinct points and can be parallel to at most one. So, it must contain at least 2 points of $S$. Consider a line $l$, passing through one of the points, say $P$. If $l$ is not parallel to $QR$, then it meets $QR$ in a point of $S$. Finally, suppose $l$ passes through $P$ and is parallel to $QR$. The unique line through $Q$ parallel to $PR$ must meet $l$ in a point of $S$ other than $P$, for if it is parallel to $l$, by transitivity it would be parallel to $QR$, a contradiction. So $P$ is a projective plane, and $S = P \setminus g_\infty$. \qed
Affine Planes

As a consequence of this theorem we see that the Euclidean plane $\mathbb{R}^2$ is an affine plane. The projective closure of this affine plane is called the \textit{real projective plane} which we have met before.
Finite Affine Spaces

Def: The order of a finite affine space is the order of its projective closure.

Theorem 1.6.5: Let $A$ be a finite d-dimensional affine space of order $q$. Then

a. There exists an integer $q \geq 2$ such that any line of $A$ is incident with exactly $q$ points.

b. If $U$ is a $t$-flat, then $|U| = q^t$.

Pf: Let $A = P \setminus H_\infty$.

(a) Let $g$ be a line of $A$. As a line of $P$ we know that it contains $q+1$ ($\geq 3$) points of $P$. Since $g$ is not in $H_\infty$, it intersects $H_\infty$ in a unique point. Thus, $q$ of the points of this line are in $A$. 
Finite Affine Spaces

Theorem 1.6.5:

b. If $U$ is a $t$-flat, then $|U| = q^t$.

Pf (cont.):
(b) Any $t$-dimensional subspace $U$ of $A$ intersects – when considered as a subspace of $P$ – the hyperplane at infinity in a $(t-1)$-dimensional subspace $U \cap H_\infty$. Thus, we have:

\[
\# \text{ points of } A \text{ in } U = \# \text{ points of } U \text{ in } P - \# \text{ points in } U \cap H_\infty.
\]

\[
= q^t + \ldots + q + 1 - (q^{t-1} + \ldots + q + 1) = q^t.
\]

\[\square\]
Finite Affine Spaces

**Corollary 1.6.6:** The total number of points in a finite affine plane of order $q$ is $q^2$ and the number of points on a line is $q$. 
Finite Affine Planes

Theorem 1.6.7: If $S$ is a rank 2 geometry satisfying:

a. 2 points determine a unique line,

b. there are $q^2$ points in total ($q \geq 2$), and

c. each line has exactly $q$ points.

Then $S$ is an affine plane.

\textit{Pf:} We will show that $S$ satisfies conditions (1), (2) and (3) of Corollary 1.6.3, so Theorem 1.6.4 will give the result. As (1) and (a) are identical, and (3) follows from (b) and (c) since $q^2 > q$, we need only prove (2).

First we show that each point of $S$ is on $q+1$ lines. Let $r$ be the number of lines through a point $P$. The $q^2 - 1$ points of $S$ other than $P$ are distributed $q-1$ at a time on each of these lines. Thus $r = \frac{q^2 - 1}{q - 1} = q + 1$. 
Finite Affine Planes

Theorem 1.6.7: If $S$ is a rank 2 geometry satisfying:

a. 2 points determine a unique line,

b. there are $q^2$ points in total ($q \geq 2$), and

c. each line has exactly $q$ points.

Then $S$ is an affine plane.

Proof (cont.): Now consider a non-incident point line pair $(P, g)$ [an anti-flag]. Since $g$ has exactly $q$ points, the point $P$ is joined to the points of $(g)$ by $q$ lines in total. Hence, there remains exactly one line through $P$ which has no point in common with $g$. 

\[\square\]