Pairwise Balanced Designs
Definition

We have defined these before, but now refine the definition.

A \((v,K,\lambda)\)-pairwise balanced design (or \((v,K,\lambda)\)-PBD) where \(v \geq 2\), \(\lambda \geq 1\) and \(K \subseteq \{n \in \mathbb{Z} \mid n \geq 2\}\) is a set system \((X, \mathcal{A})\) such that:

1) \(|X| = v\),
2) \(|A| \in K\) for all \(A \in \mathcal{A}\), and
3) every pair of distinct points is contained in exactly \(\lambda\) blocks.

Recall that a PBD is allowed to have blocks of size \(v\). Also, we have that a \((v,k,\lambda)\)-BIBD is an \((v,\{k\},\lambda)\)-PBD and conversely, a \((v,\{k\},\lambda)\)-PBD is a \((v,k,\lambda)\)-BIBD provided \(k < v\).
Some Constructions

If there exists a TD(k+1,t) with \(k \geq 2\) then the following PBDs exist:

1) a \((kt+u, \{k,k+1,t,u\}, 1)\)-PBD for all \(u\) with \(2 \leq u \leq t-1\),
2) a \((kt+1, \{k,k+1,t\}, 1)\)-PBD, and
3) a \(((k+1)t, \{k+1,t\}, 1)\)-PBD.

\textbf{Pf:} For 1) remove \(t-u\) points from one groop of a TD(k+1,t) and take all the groops and blocks of the truncated design as blocks of the PBD.

For 2) remove \(t-1\) points from one groop of a TD(k+1,t) and take all the groops (except the one with 1 point) and blocks of the truncated design as blocks of the PBD.

For 3) take all the blocks and groops of the TD as blocks of the PBD.
**Some Constructions**

**Lemma**: If there exists a resolvable \((v,k,1)\)-BIBD then there exists a \((v+r, \{k+1,r\},1)\)-PBD where \(r = (v-1)/(k-1)\).

**Pf.** Add a new and distinct point to each block of each parallel class and one new block containing all the new points.

Note that if we start with an affine plane this construction will produce a BIBD since \(r = k+1\) in this case (and the result is a projective plane).

**Corollary**: For all even \(v \geq 4\) \(\exists (2v-1, \{3,v-1\},1)\)-PBD.  
**Pf:** Apply lemma with \(k = 2\) and recall that resolvable \((v,2,1)\)-BIBDs are complete graphs with 1-factors.
Numerical Conditions

Throughout the rest of this chapter we will assume that:

K is a subset of at least two integers.

Define

$$B(K) = \{v \mid \text{there exists a } (v,K,1)\text{-PBD}\}.$$ 

Furthermore, define

$$\alpha(K) = \gcd\{ k-1 \mid k \in K\}$$

and

$$\beta(K) = \gcd\{ k(k-1) \mid k \in K\}.$$ 

Note that $$K \subseteq B(K)$$ since a $$(k, \{k\}, 1)$$-PBD exists for all $$k \geq 2$$. (One block of size $$k$$ ... allowable for PBD but not for BIBD.)
Lemma: Let $v \geq 3$ be an integer. $v$ in $B(K)$ only if

$v - 1 \equiv 0 \pmod{\alpha(K)}$

and

$v(v-1) \equiv 0 \pmod{\beta(K)}$.

Pf: Suppose $v \in B(K)$, then there exists a $(v,K,1)$-PBD. Let $X$ be the set of points of this design and $x \in X$. Let $r' = r(x)$ denote the number of blocks containing $x$ and denote these blocks by $A_1, ..., A_{r'}$. Then

$$\sum_{i=1}^{r'} (|A_i| - 1) = v - 1.$$ 

Since $|A_i| - 1 \equiv 0 \pmod{\alpha(K)}$ for all $i$ we get the first condition. The second follows from considering all blocks.
Example

Suppose $K = \{3,4,6\}$, then
\[
\alpha(K) = \gcd \{2,3,5\} = 1 \quad \text{and} \\
\beta(K) = \gcd \{6,12,30\} = 6.
\]

According to the lemma, necessary conditions for the existence of a PBD with block sizes 3, 4 or 6 are that
\[
v - 1 \equiv 0 \pmod{1} \quad \text{and} \\
v(v-1) \equiv 0 \pmod{6}.
\]

The first condition just says that $v$ is an integer. The second condition is satisfied iff $v(v-1) \equiv 0 \pmod{3}$, so we have $v \equiv 0 \text{ or } 1 \pmod{3}$. $v \geq 3$ follows since 3 is the smallest block size.
PBD Closure

K is said to be *PBD-closed* if \( B(K) = K \), i.e., if there exists a \((v, K, 1)\)-PBD, then \( v \in K \).

**Theorem**: \( B(K) \) is PBD-closed, that is, \( B(B(K)) = B(K) \).

**Pf**: Assume \( v \in B(B(K)) \). Let \((X, \mathcal{A})\) be any \((v, B(K), 1)\)-PBD. For each \( A \in \mathcal{A} \) there is an \((|A|, K)\) – PBD, say \((A, \mathcal{B}_A)\) [this follows from the definition of \( B(K) \)]. Define \( \mathcal{B} = \bigcup_{A \in \mathcal{A}} \mathcal{B}_A \).

Now \((X, \mathcal{B})\) is a \((v, K, 1)\)-PBD, so \( v \in B(K) \) and we have \( B(K) \supseteq B(B(K)) \). Since \( L \subseteq B(L) \) we have \( B(K) \subseteq B(B(K)) \) giving the result.

Rick Wilson refers to this result as “Breaking up Blocks”.

PBD Closure

For an integer $k \geq 2$ define
$$V_k = \{k\} \cup \{v \mid \text{there is a } (v,k,1)-\text{BIBD}\}.$$ 

**Corollary:** $V_k$ is PBD-closed.

**Pf:** Note that $B(\{k\}) = V_k$ (when $v = k$ we have a PBD, otherwise it is a BIBD). Now apply the previous corollary with $K = \{k\}$. 
Example

By construction 3 at the start of this section, a TD(3,7) yields a (21, {3,7},1)-PBD. Since 3 and 7 are in $V_3$, we have that 21 is in $\mathcal{B}(V_3)$. By the Corollary, this means that 21 is in $V_3$. So, there exists a (21,3,1)-BIBD. The proof also indicates how we will construct this BIBD. The PBD has 49 blocks of size 3 and three blocks of size 7. For each block $A$ of size 7, replace the block by 7 blocks of a $(7,3,1)$-BIBD on the point set $A$ [here we are using the fact that 7 is in $V_3$]. We have thus constructed a $(21, \{3\}, 1)$-PBD which is a $(21,3,1)$-BIBD.
Groop Divisible Designs

To get another PBD-closure result we need another definition.

A *groop-divisible design* (GDD) is a triple \((X, \mathcal{G}, \mathcal{A})\) such that:

1) \(X\) is a finite set of elements called *points*,
2) \(\mathcal{G}\) is a partition of \(X\) into at least two nonempty subsets called *groops* (note that groops of size 1 are allowed),
3) \(\mathcal{A}\) is a set of subsets of \(X\) called *blocks* such that \(|\mathcal{A}| \geq 2\), for all \(A\) in \(\mathcal{A}\),
4) a groop and a block contain at most one common point, and
5) every pair of points from distinct groops is contained in exactly one block.
Transversal designs are examples of GDDs.

**Lemma**: If \((X, \mathcal{G}, \mathcal{A})\) is a GDD, then \((X, \mathcal{B})\) is a PBD with \(\lambda = 1\), where
\[
\mathcal{B} = \mathcal{A} \cup \{G \in \mathcal{G} \mid |G| \geq 2\}.
\]

**Lemma**: If \((X, \mathcal{G}, \mathcal{A})\) is a GDD, \(\infty \notin X\), then with
\[
\mathcal{B} = \mathcal{A} \cup \{G \cup \{\infty\} \mid G \in \mathcal{G}\},
\]
\((X \cup \{\infty\}, \mathcal{B})\) is a PBD with \(\lambda = 1\).

**Lemma**: If \((X, \mathcal{A})\) is a PBD with \(\lambda = 1\), then \((X, \mathcal{G}, \mathcal{A})\) is a GDD where
\[
\mathcal{G} = \{\{x\} \mid x \in X\}.
\]
GDDs and BIBDs

Lemma: Suppose that $v > k > 1$. There exists a $(v,k,1)$-BIBD if and only if there exists a GDD with $v-1$ points, $r$ groops of size $k-1$ and blocks of size $k$ 

$$r = \frac{v-1}{k-1}.$$ 

Pf: Choose any point, $x$, of the $(v,k,1)$-BIBD. Define as groops the elements of the blocks of the BIBD which contain $x$, without $x$ itself. The remaining blocks will be blocks of the GDD.

The converse follows from the second construction of the last page. Note that all the blocks of the PBD will have size $k$, so the design is a BIBD.
Another PBD-closed set

For any $k \geq 2$ define

$$R_k = \{r \mid \text{there exists an } (r(k-1)+1, k, 1)\text{-BIBD}\}.$$ 

**Theorem:** $R_k$ is PBD-closed.

**Pf:** Let $(X, \mathcal{A})$ be any $(v,R_k)$-PBD. For each block $A \in \mathcal{A}$, there is an $(|A|(k-1)+1,k,1)$-BIBD [by the definition of $R_k$]. By the previous lemma, this is equivalent to a GDD with $|A|(k-1)$ points, $|A|$ groups of size $k-1$, and blocks of size $k$. We may construct this GDD in the following way. Let $J$ be some set of size $k-1$ and take as a point set for the GDD the set $A \times J$. The groups will be the sets $\{x\} \times J$ for $x \in A$, and let $\mathcal{B}_A$ denote the set of blocks of the GDD.
Another PBD-closed set

**Theorem**: \( R_k \) is PBD-closed.

**Pf**: (cont) Let \( Y = X \times J \), \( \mathcal{H} = \{ \{x\} \times J \mid x \in X \} \), and
\[
\mathcal{B} = \bigcup_{A \in \mathcal{A}} \mathcal{B}_A.
\]
We will show that \((Y, \mathcal{H}, \mathcal{B})\) is a GDD with \( v(k-1) \) points, \( v \) groops of size \( k-1 \) and blocks of size \( k \). Then, by the previous lemma there exists a \((v(k-1)+1,k,1)\)-BIBD and so \( v \in R_k \) thus showing that \( R_k \) is PBD-closed.

\((Y, \mathcal{H}, \mathcal{B})\) certainly has \( v(k-1) \) points, \( v \) groops of size \( k-1 \) and blocks of size \( k \). Consider two points from different groops, \((x,i)\) and \((y,j)\) with \( x \neq y \). There is a unique block \( A \) in \( \mathcal{A} \) containing \( x \) and \( y \). There will then be a unique block \( B \) in \( \mathcal{B}_A \) containing \( x \) and \( y \). \( B \) is then the unique block in \( \mathcal{B} \) containing these two points, so \((Y, \mathcal{H}, \mathcal{B})\) is a GDD.
Example

Let $k = 3$. There is a well known $(7,3,1)$-BIBD, and since $7 = 3(3-1) + 1 = 3(k-1)+1$, we have $3 \in R_3$. Then, because a $(7,3,1)$-BIBD exists and $R_3$ is PBD-closed, $7 \in R_3$, which means that a $(15,3,1)$-BIBD exists since $15 = 7(3-1)+1$.

We can construct this BIBD by following the proof. Start with the $(7,3,1)$-BIBD where $\mathcal{A} = \{124, 235, 346, 457, 156, 267, 137\}$. The GDD equivalent to this (obtained by removing 7) has groups 13, 26, 45 and blocks 124, 235, 346 and 156.

We will now rewrite this GDD with new elements. Take $J = \{0,1\}$ and $A = \{x,y,z\}$ and write the elements of $A \times J$ as follows:

1 = (x,0) = x_0  
2 = (y,0) = y_0  
4 = (z,0) = z_0  
3 = (x,1) = x_1  
6 = (y,1) = y_1  
5 = (z,1) = z_1

Blocks $\mathcal{B}_{xyz} := x_0y_0z_0 \ x_1y_0z_1 \ x_1y_1z_0 \ x_0y_1z_1$
Example

Now, letting each block of $A$ be $xyz$ in turn, we replace the block by the 4 blocks of $B_{xyz}$. Thus, for example we get the blocks:

124 $\rightarrow$ 10200 12041 12401 10241

137 $\rightarrow$ 10370 13710 13701 10371

etc. This gives 28 blocks on 14 points. There are 7 groups of size 2, and we form new blocks by adding a new point to each of these. Some of these new blocks would look like: $\infty101$, $\infty202$, $\ldots$, $\infty707$

We now have 35 blocks on 15 points all of size 3, i.e., a (15,3,1)-BIBD.
We return to the Steiner Triple Systems (\((v,3,1)\)-BIBDs) which we have shown exist iff \(v \equiv 1 \text{ or } 3 \mod 6, \ v \geq 7\) and establish this result again by what are called recursive construction techniques (i.e., PBD closure). The general steps that we use for STS's can be used in other situations.

Since \(r = (v-1)/2\) in this case \((k = 3)\), we can express the necessary conditions on \(v\) in terms of \(r\). Thus, \(R_k = R_3 = \{r| \text{ a } (v, 3, 1) \text{ -BIBD exists}\}\) and

\[
\begin{align*}
v &\equiv 1 \mod 6 \rightarrow r \equiv 0 \mod 3 \\
v &\equiv 3 \mod 6 \rightarrow r \equiv 1 \mod 3, \text{ so} \\
R_3 &\subseteq \{r| r \equiv 0 \text{ or } 1 \mod 3, r \geq 3\}.
\end{align*}
\]
STS's again

The first step is to find (by any means) some small values that are in \( R_3 \).

3 \( \varepsilon R_3 \) since a (7,3,1)-BIBD exists. \([3 = (7-1)/2]\)

4 \( \varepsilon R_3 \) since a (9,3,1)-BIBD exists. This is the affine plane of order 3.

6 \( \varepsilon R_3 \) if a (13,3,1)-BIBD exists. We shall construct one.

In \( \mathbb{Z}_{13} \) consider the “base” blocks \{0,1,4\} and \{0,2,8\}. The differences from the first block are 1,4,12,3,9 and 10, while those from the second are 2,8,11,6,5 and 7. Since all (non-zero) differences occur and none are repeated, when these two blocks are developed, no pair of elements in one development can appear in the other development.

\( \{3,4,6\} \subseteq R_3 \)
STS's again

The second step will be to determine what $B\{3,4,6\}$ is. We start by showing that certain numbers are in it.

$7 \in B\{3\} \subseteq B\{3,4,6\}$ since a $(7,3,1)$-BIBD exists.

$18 \in B\{3,6\}$ since a TD$(3,6) \rightarrow (18,\{3,6\},1)$-PBD exists.

[A TD$(3,n)$ always exists since you only need 1 Latin square of order $n$ to construct one.]

$19 \in B\{3\}$. Adjoin a new point to the groups of a TD$(3,6)$ to obtain a $(19,\{3,7\},1)$-PBD. Since 3 and 7 are in $B\{3\}$ and $B\{3\}$ is PBD closed, $19 \in B\{3\}$.

$\{7,18,19\} \subseteq B\{3,4,6\}$
STS's again

To complete the determination we need the following construction.

**Lemma:** If \( t \equiv 0 \text{ or } 1 \mod 3, \ t \geq 3 \text{ and } t \neq 6 \) then the following PBD's exist in which all block sizes are \( \equiv 0 \text{ or } 1 \mod 3 \).

1) If \( u \equiv 0 \text{ or } 1 \mod 3 \text{ and } 3 \leq u \leq t \), then there exists a \((3t+u, \{3,4,t,u\},1)\)-PBD, and

2) If \( u \in \{0,1\}, \) then there exists a \((3t+u, \{3,4,t\},1)\)-PBD.

**Pf:** A TD(4,t) exists provided \( t \neq 2,6 \) since it is equivalent to a pair of orthogonal Latin squares. The constructions at the start of the chapter then give these results.
Theorem: $\mathcal{B}({3,4,6}) = \{n \geq 3 \mid n \equiv 0,1 \mod 3 \}$.

Pf: $\alpha({3,4,6}) = 1$ and $\beta({3,4,6}) = \gcd(6,12,30) = 6$ and as shown in a previous example we must have $\mathcal{B}({3,4,6}) \subseteq \{n \geq 3 \mid n \equiv 0,1 \mod 3 \}$.

We show equality by constructions, and we will provide the constructions by induction.

Throughout this proof we shall only consider $v$'s for which $v \equiv 0,1 \mod 3$ and $v \geq 3$. As base cases we can take $v = 3,4$ or 6. Consider a specific $v_0$. Our induction hypothesis is that $v \in \mathcal{B}({3,4,6})$ for all $3 \leq v < v_0$. We can assume that $v_0 \geq 7$ and we want to show that $v_0 \in \mathcal{B}({3,4,6})$.

Write $v_0 = 9s + j$ where $j \in \{0,1,3,4,6,7\}$. 
Theorem: \( B(\{3,4,6\}) = \{n \geq 3 \mid n \equiv 0,1 \mod 3 \} \).

Pf: Consider the following table:

<table>
<thead>
<tr>
<th>( v_0 )</th>
<th>conditions</th>
</tr>
</thead>
<tbody>
<tr>
<td>3t + u</td>
<td></td>
</tr>
<tr>
<td>9s</td>
<td>3(3s) ( t \neq 6 \rightarrow s \neq 2 )</td>
</tr>
<tr>
<td>9s + 1</td>
<td>3(3s) + 1 ( t \neq 6 \rightarrow s \neq 2 )</td>
</tr>
<tr>
<td>9s + 3</td>
<td>3(3s+1)</td>
</tr>
<tr>
<td>9s + 4</td>
<td>3(3s+1) + 1</td>
</tr>
<tr>
<td>9s + 6</td>
<td>3(3s+1) + 3</td>
</tr>
<tr>
<td>9s + 7</td>
<td>3(3s+1) + 4</td>
</tr>
</tbody>
</table>

Now \( t \geq 3 \) provided \( s \geq 1 \). \( t \equiv 0,1 \mod 3 \) in all cases, and \( t \neq 6 \) except as noted. So, unless \( v_0 \) is 7, 18 or 19 the conditions of the lemma are met and we can construct a \((v_0, \{3,4,t\},1)\)-PBD. Now \( t \) is of the correct form and \( t < v_0 \)
**Theorem:** \( B(\{3,4,6\}) = \{ n \geq 3 \mid n \equiv 0,1 \text{ mod } 3 \} \).

**Pf:** so by the induction hypothesis, \( t \in B(\{3,4,6\}) \) and thus, by PBD closure, \( v_0 \in B(\{3,4,6\}) \). We have already shown that the three excepted values of \( v_0 \) are in this set, so the proof is completed.

Step 3 just puts it all together using PBD-closure.

**Theorem:** There exists an STS(\( v \)) iff \( v \equiv 1,3 \text{ mod } 6, \ v \geq 7 \).

**Pf:** \( \{3,4,6\} \subseteq R_3 \rightarrow B(\{3,4,6\}) \subseteq B(R_3) = R_3 \), so
\[
\{ n \geq 3 \mid n \equiv 0,1 \text{ mod } 3 \} \subseteq R_3 \subseteq \{ n \geq 3 \mid n \equiv 0,1 \text{ mod } 3 \}.
\]
We will repeat this process for BIBDs with block size 4 and $\lambda = 1$.

First we note that since $r = (v-1)/(4-1)$ must be an integer, $3 | v-1$, i.e., $v \equiv 1 \mod 3$, and since $b$ is an integer $12 | v(v-1)$. This means that either $4 | v$ or $4 | v-1$. So, either $v \equiv 4 \mod 12$ or $v \equiv 1 \mod 12$, and since $v > k$, $v \geq 13$. This translates to $r \equiv 0,1 \mod 4$ and $r \geq 4$. Thus, $R_4 \subseteq \{n \geq 4 \mid n \equiv 0, 1 \mod 4\}$.

Step 1: Find some small values in $R_4$.

$4 \in R_4$: The projective plane of order 3 has $r = 4$.

$5 \in R_4$: The affine plane of order 4 is a $(16,4,1)$-BIBD.
(v,4,1) – BIBDs

8 ∈ R₄ : In (ℤ₅ x ℤ₅, +) develop the two base blocks
{(0,0),(0,1),(1,0),(2,2)} and {(0,0),(0,2),(2,0),(4,4)}
to get a (25,4,1)-BIBD.

9 ∈ R₄ : In (ℤ₃ x ℤ₃ x ℤ₃, +) develop the two base blocks
{000,020,111,211} and {000,102,012,110}. Form
groups Gₓ,y = {xyz | z ∈ ℤ₃}. This is a GDD with
9 groups of size 3 and blocks of size 4. Adding ∞
to each group yields a (28,4,1)-BIBD.

12 ∈ R₄ : Develop the 3 blocks in (ℤ₃₇,+),
{0,1,3,24}, {0,10,18,30}, {0,4,26,32}
to obtain a (37,4,1)-BIBD.
Step 2 is to show that
\[ \mathcal{B}(\{4,5,8,9,12\}) = \{n \geq 4 \mid n \equiv 0, 1 \mod 4\}. \]

Pf: \( \alpha(\{4,5,8,9,12\}) = 1 \) and \( \beta(K) = \gcd(12,20,56,72,132) = 4 \)
and so
\[ \mathcal{B}(\{4,5,8,9,12\}) \subseteq \{n \geq 4 \mid n \equiv 0, 1 \mod 4\}. \]
We show equality by constructions, and we will provide the constructions by induction.
Throughout this proof we shall only consider \( v \)'s for which
\( v \equiv 0, 1 \mod 4 \) and \( v \geq 4 \). As base cases we can take
\( v = 4,5,8,9 \) or 12. Consider a specific \( v_0 \). Our induction hypothesis is that
\( v \in \mathcal{B}(K) \) for all \( 4 \leq v < v_0 \). We can assume that \( v_0 \geq 13 \) and we want to show that \( v_0 \in \mathcal{B}(K) \).
Write \( v_0 = 48s + j \) where \( j \equiv 0,1 \mod 4 \).
(v,4,1) - BIBDs

We will construct infinite families by truncating TD(5,t) designs. As these are equivalent to 3 MOLS(t), we can use McNeish's result to say that these will exist unless 2 but not 4 divides t or 3 but not 9 divides t. Thus we must avoid those t's for which \( t \equiv 0, 2, 3, 6, 10 \mod 12 \). To carry out the truncation we will be removing \( t-u \) points from a group of the TD(5,t), so we will also require \( 0 \leq u \leq t - 1 \).

The following table shows how to write \( v_0 \) in the form \( 4t+u \) and gives the requirements needed to satisfy the above constraints.
\((v,4,1)\)-BIBDs

<table>
<thead>
<tr>
<th>(v_0)</th>
<th>4(t+u)</th>
<th>restriction</th>
<th>(v_0)</th>
<th>4(t+u)</th>
<th>restriction</th>
</tr>
</thead>
<tbody>
<tr>
<td>48s</td>
<td>4(12s-4)+16</td>
<td>(s \geq 2)</td>
<td>48s+24</td>
<td>4(12s+5)+4</td>
<td></td>
</tr>
<tr>
<td>48s+1</td>
<td>4(12s-4)+17</td>
<td>(s \geq 2)</td>
<td>48s+25</td>
<td>4(12s+5)+5</td>
<td></td>
</tr>
<tr>
<td>48s+4</td>
<td>4(12s+1)</td>
<td></td>
<td>48s+28</td>
<td>4(12s+5)+8</td>
<td>(s \geq 1)</td>
</tr>
<tr>
<td>48s+5</td>
<td>4(12s+1)+1</td>
<td></td>
<td>48s+29</td>
<td>4(12s+5)+9</td>
<td>(s \geq 1)</td>
</tr>
<tr>
<td>48s+8</td>
<td>4(12s+1)+4</td>
<td></td>
<td>48s+32</td>
<td>4(12s+8)</td>
<td></td>
</tr>
<tr>
<td>48s+9</td>
<td>4(12s+1)+5</td>
<td></td>
<td>48s+33</td>
<td>4(12s+8)+1</td>
<td></td>
</tr>
<tr>
<td>48s+12</td>
<td>4(12s+1)+8</td>
<td></td>
<td>48s+36</td>
<td>4(12s+8)+4</td>
<td></td>
</tr>
<tr>
<td>48s+13</td>
<td>4(12s+1)+9</td>
<td>(s \geq 1)</td>
<td>48s+37</td>
<td>4(12s+8)+5</td>
<td></td>
</tr>
<tr>
<td>48s+16</td>
<td>4(12s+4)</td>
<td></td>
<td>48s+40</td>
<td>4(12s+8)+8</td>
<td>(s \geq 1)</td>
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<tr>
<td>48s+17</td>
<td>4(12s+4)+1</td>
<td></td>
<td>48s+41</td>
<td>4(12s+8)+9</td>
<td>(s \geq 1)</td>
</tr>
<tr>
<td>48s+20</td>
<td>4(12s+5)</td>
<td></td>
<td>48s+44</td>
<td>4(12s+8)+12</td>
<td>(s \geq 1)</td>
</tr>
<tr>
<td>48s+21</td>
<td>4(12s+5)+1</td>
<td></td>
<td>48s+45</td>
<td>4(12s+8)+13</td>
<td>(s \geq 1)</td>
</tr>
</tbody>
</table>

Restrictions due to the \(0 \leq u \leq t - 1\) condition.
(v,4,1)-BIBDs

The truncated TD(5,t) that results is a
(v₀, {4,5,8,9,12,t,u},1) – PBD with t ≡ 0,1 mod 4 and
4 ≤ u ≤ 17 or a (v₀, {4,5,8,9,12,t},1)-PBD if u = 0,1.

Since t is of the correct form, t < v₀ and t ≥ 4 (under the
restrictions), we have t in \( B\)({4,5,8,9,12}) by induction. If
we have u = 13, 16 or 17 then we have u \( \in \ B\)({4,5}).
13 \( \in \ B\)({4,5}): PG(2,3) is a (13,4,1)-BIBD.
16 \( \in \ B\)({4,5}): AG(2,4) is a (16,4,1) – BIBD.
17 \( \in \ B\)({4,5}): Starting with PG(2,4) which is a (21,5,1)-
BIBD, remove 4 points from one line. This leaves 17
points and all blocks have either 4 or 5 points. There are
20 blocks in the resulting PBD.
We now only have to deal with the exceptional cases to prove the result.

13 $\varepsilon$ W : A projective plane of order 3 is a $(13,4,1)$-BIBD, and so, a $(13,\{4\},1)$-PBD.

28 $\varepsilon$ W : A $(28,4,1)$-BIBD was constructed to show that $9 \in R_4$.

29 $\varepsilon$ W : A TD$(4,7)$ with a new point added to each group gives a $(29,\{4,8\},1)$-PBD.

40 $\varepsilon$ W : A PG$(3,3)$ is a $(40,4,1)$-BIBD.

41 $\varepsilon$ W : Truncate 4 points from a group of a TD$(5,9)$ to get a $(41,\{4,5,9\},1)$-PBD.

44 $\varepsilon$ W : Truncate 1 point from a group of a TD$(5,9)$ to get a $(44,\{4,5,8,9\},1)$-PBD.
(v,4,1)-BIBDs

45 \varepsilon W: A TD(5,9) gives a (45,\{5,9\},1)-PBD.
48 \varepsilon W: A TD(4,12) gives a (48,\{4,12\},1)-PBD.
49 \varepsilon W: A TD(4,12) with a new point added to each group gives a (49,\{4,13\},1)-PBD. Since there is a (13,\{4\},1)-PBD [projective plane of order 3], there exists a (49,\{4\},1)-PBD by PBD closure.

And Step 3 puts it all together.

**Theorem:** There exists a (v,4,1)-BIBD if and only if \( v \equiv 1 \text{ or } 4 \mod 12 \) and \( v \geq 13 \).

**Pf:** \( \{4,5,8,9,12\} \subseteq R_4 \rightarrow B(\{4,5,8,9,12\}) \subseteq B(R_4) = R_4 \), so \( \{n \geq 4 \mid n \equiv 0,1 \mod 4\} \subseteq R_4 \subseteq \{n \geq 4 \mid n \equiv 0,1 \mod 4\} \).
The Reverend Thomas Penyngton Kirkman (1806-1895), vicar of the Parish of Southworth, Lancashire and mathematician. Most famous for a minor contribution of his which appeared as Query 6 on page 48 of the Lady’s and Gentlemen’s Diary of 1850.

**Fifteen young ladies of a school walk out three abreast for seven days in succession: it is required to arrange them daily so that no two shall walk abreast more than once.**
Kirkman Triple Systems

The solution to Kirkman’s 15 schoolgirl problem is a resolution of a block design on 15 points with blocks of size 3 such that every pair of points is contained in a unique block. These block designs are, of course, Steiner Triple Systems (on 15 points).

A resolvable Steiner Triple System is called a Kirkman Triple System and there are 7 non-isomorphic Kirkman Triple Systems on 15 points.

We can use the same techniques of PBD-closure to prove that there exists a Kirkman Triple System of order $v$ if and only if $v \equiv 3 \mod 6$ and $v \geq 9$. 