The Ubiquitous Translation Hyperoval Revisited

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A translation oval $\Omega$ with axis $\ell$ in a projective plane $\pi$ of order $n$ is an oval with tangent line $\ell$ such that the affine plane $\pi \setminus \ell$ admits a translation group $T$ having the affine points of $\Omega$ as a $T$-orbit.

- $n = 2^e$
- $T$ is an elementary abelian 2-group of order $n$
- every involution of $T$ has a different center
- there is a unique point $N$ on $\ell \setminus \Omega$ which is not the center of an involution in $T$. 

Translation Hyperovals
Translation Hyperovals

Ω ∪ {N} is called a translation hyperoval with axis \( l \).

If \( P \) is the point of tangency, then \{P, N\} is called the carrier set of the translation hyperoval.

Note that \{Ω - \{P\}\} ∪ {N} is also a translation oval.

In PG(2,2^e), \( Ω \) and this other translation oval are not projectively equivalent.

“This is too fussy a definition of equivalence”
Translation Hyperovals

A translation oval having more than one axis is a conic, and all the tangent lines are axes.

A conic together with it's nucleus is called a hyperconic.

Note that each “pointed conic” (remove a point of the conic and add the nucleus) is a translation oval which is not a conic.

A translation oval not contained in a hyperconic is called proper.
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Let $S$ be a spread in $\mathcal{P} = \text{PG}(2N-1,2)$, consisting of $2^N + 1$ pairwise skew subspaces, each of dimension $N - 1$. An $N-1$ dimensional subspace $\tau$ of $\mathcal{P}$ is a transversal to $S$, relative to the carrier set $\{X,Y\} \subset S$, if

$$|\gamma \cap \tau| = 1, \ \forall \ \gamma \in S - \{X,Y\},$$

$$|\gamma \cap \tau| = 0 \text{ if } \tau \in \{X,Y\}.$$

Thm: The translation plane $\pi_s$, associated with $S$, admits a translation oval relative to the natural axis precisely when $S$ admits a transversal.
This can be rephrased in terms of spread sets.

A collection $\tau^*$ of $2^N + 1$ distinct $N \times N$ matrices over $F_2$ is a translation spread set if $\tau^* = \tau \cup \{A\}$ is such that:

a) $\tau$ is an $F_2$ spread set; i.e., $O, I \in \tau$ and the difference between any two distinct elements of $\tau$ is non-singular; and

b) $A$ is a non-singular matrix such that $\text{rank}[A-M] = 1$, $\forall M \in \tau$. 
Consider the N-1 dimensional subspace given by
\[ \theta_M = \{(x, x^r): x \in \mathbb{F}_2^N, r = 2^M\}. \]

If \( \theta_M \) is a transversal of a spread S, then \( \theta_M \) is called a \( \lambda \)-conic (with carrier set \{X,Y\}, the X and Y-axis of S).
A transversal of a Desarguesian spread which remains a transversal of a spread obtained from the Desarguesian spread by replacement, gives rise to a translation oval of the new plane which is said to be inherited from the Desarguesian plane.

Most examples of translation ovals in non-Desarguesian planes are of this inherited type.

\( \lambda \)-conics are inherited.
Several results are obtained concerning when generalized André spreads admit $\lambda$-conics.

Some of these in particular,

- If $N$ is a prime power every generalized André spread admits $\lambda$-conics.
- If $N = a$ non-prime there exist non-Desarguesian André spreads which admit $\lambda$-conics.
- If $N$ is even, but not a power of 2, there also exist André spreads which do not admit $\lambda$-conics.
The Ubiquity Papers

A characterization of which generalized André planes admit $\lambda$-conics is also given.

A transversal cover of a spread $S$ in $\mathcal{P} = \text{PG}(2N-1,2)$, relative to a pair of distinct components $\{X,Y\}$, is a collection $\Theta$ of pairwise skew transversals to $S$ such that:

a) Each transversal in $\Theta$ is skew to both $X$ and $Y$;

b) $\bigcup \Theta = \bigcup \Sigma$, where $\Sigma = S - \{X,Y\}$.
A transversal cover $\Theta$ of $S$, when it exists, is a replacement for the partial spread $\Sigma \subset S$, and we say that the replaced spread $T = \Theta \cup \{X,Y\}$ is ovally derived from $S$.

Thm: A spread is ovally derived from a Desarguesian spread if and only if it is a generalized André spread that admits $\lambda$-conics.
Subsequently it is shown that:

A nearfield plane is ovally derived from a Desarguesian plane if and only if it arises from a “strong Dickson” pair.

A Hall plane of even order $q^2$ is ovally derived from a Desarguesian plane if and only if $q$ is a square.
λ-conics

• Denniston (1979)
  Demonstrated the existence of translation ovals in non-Desarguesian planes by constructing some λ-conics (with M = 1) in some André planes.

• Glynn & Steinke (1993)
  Proved the existence of λ-conics (with M = 1) in even order Hall planes, H(q²), when q is a square.

Why doesn't everybody read my papers?
A Hall plane $H(q^2)$ is constructed from $PG(2,q^2)$ by a process called **derivation**. (A special type of subspread replacement).

This requires a **derivation set** $D$ – a set of $q+1$ points on $\ell_\infty$ such that the set of all subplanes of order $q$ (**Baer subplanes**) that contain $D$ has the property that for any two points of the affine plane that are on a line with slope in $D$ are contained in one of these subplanes.
Hall Planes

To obtain the Hall plane with respect to a derivation set D, we remove all the lines of PG(2, q^2) which intersect D (this includes ℓ∞) and declare all the Baer subplanes which belong to D to be lines. This produces an affine plane which we extend to a projective plane in the usual way. The resulting Hall plane is non-Desarguesian iff q > 2.

*Note:* Collineations of PG(2, q^2) which preserve D will also be collineations of H(q^2).
Other Translation Hyperovals

Thm: Any translation hyperoval with axis the infinite line of a Hall plane of even order having one, and only one, point of it's carrier set in the derivation set, is an inherited translation hyperoval.

- Korchmáros (1986) – gave examples of hyperconics of this type.
- O'Keefe, Pascasio, Penttila (1992) – established the result for all hyperconics of this type.
- WEC (unpublished) – extended the result to all translation hyperovals.
Classification of Hyperconics

Thm: In Hall planes of even order, $H(q^2)$, the only inherited hyperconics are those with carrier set on the infinite line, with one and only one of the carrier points in the derivation set and if $q$ is a square, the Glynn-Steinke hyperconics ($\lambda$-conics).

- O'Keefe & Pascasio (1996) – started this classification by showing that hyperconics with both carrier points in the derivation set do not inherit.
- WEC (2009?) - finished the classification by showing that there are no inherited hyperconics in the remaining cases.
A complete search for hyperovals in Hall(16) was made in 1991, but the table reporting the results was incorrect.

Hyperovals of Hall(16) containing the origin

<table>
<thead>
<tr>
<th>Type</th>
<th>Class</th>
<th>Number</th>
<th>Group</th>
</tr>
</thead>
<tbody>
<tr>
<td>H1 (10)</td>
<td>---</td>
<td>None</td>
<td>-----------</td>
</tr>
<tr>
<td>H2 (60)</td>
<td>I</td>
<td>15</td>
<td>T sd $Z_4$</td>
</tr>
<tr>
<td></td>
<td>II</td>
<td>15</td>
<td>T sd $Z_4$</td>
</tr>
<tr>
<td>H3 (60)</td>
<td>I</td>
<td>60</td>
<td>T</td>
</tr>
<tr>
<td>H4 (6)</td>
<td>I</td>
<td>30</td>
<td>T sd $(Z_5 \text{ sd } Z_4)$</td>
</tr>
</tbody>
</table>
The hyperovals in this plane:

H1 – (none) two infinite points in the derivation set.

H2 – O'Keefe-Pascasio-Penttila
the two classes correspond to which carrier point is in the derivation set.

H3 – Non-inherited

H4 – Glynn-Steinke (6 pairs of conjugate infinite points)
Non-inherited Hyperovals

We will restrict ourselves to PG(2,16), but the construction technique (alas, not the proof) most likely generalizes to higher orders. Throughout this section $q = 4$.

Let $\eta$ be an element of norm 1 other than 1. ($\eta^{q+1} = 1$)

Consider two conics, having affine equations:
- $C_1 : Ax^2 + (\eta+1)x + y = 0$, and
- $C_2 : Ax^2 + (\eta+1)x + y + (\eta+1)/A = 0$,

where $A \neq 0$.

Note: These have the same point of tangency and nucleus.
Non-inherited Hyperovals

The idea is to pick $q^2/2$ affine points from each conic so that the resulting set has the appropriate structure.

Let $T_0$ be the set of elements of absolute trace 0 in $\text{GF}(q^2)$, and define

$$G_1 = \frac{\eta^2 + \eta + 1}{A} T_0$$

and

$$G_2 = \text{GF}(q^2) \setminus G_1$$

$S$ is constructed by taking the points of $C_1$ whose x-coordinates lie in $G_1$ and the points of $C_2$ whose x-coordinates lie in $G_2$. 
Non-inherited Hyperovals

1. $\mathcal{S}$ is a translation set. $\mathcal{S}$ is invariant under the translations $(x,y) \rightarrow (x+a,y+b)$ for every $(a,b) \in \mathcal{S}$.

2. There are $q^2/4$ lines with slope 1 which meet $\mathcal{S}$ in 4 points and all other lines meet $\mathcal{S}$ in no more than 2 points.

3. The vertical lines and lines with slope $\eta + 1/\eta$ are all tangent lines to $\mathcal{S}$.

4. The Baer subplanes belonging to $D$ meet $\mathcal{S}$ in no more than 2 points.

5. In Hall(16) $\mathcal{S}$ extends to a translation hyperoval.
Happy Birthday Norm!