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OVALS IN FIGUEROA PLANES

Dedicated to Giuseppe Tallini on the occasion of his 60th birthday

William E. Cherowitzo

A class of ovals, called the *Ovali di Roma*, is constructed in the non-Desarguesian finite Figueroa planes of odd order.

1. INTRODUCTION

An *oval* in a finite projective plane of order n is a set of $n + 1$ points, no three of which are collinear. A classical conjecture, part of the folklore in this area of mathematics, is that every projective plane contains ovals. We shall provide further evidence in support of this conjecture.

The class of non-Desarguesian planes known as the Figueroa planes was initially constructed by algebraic means (see Figueroa [3], Hering-Schaeffer [5], Dempwolff [2], and Brown [1]). The finite Figueroa planes are of order q^3, with q any prime power, $q > 2$. In the original paper [3], the value of q was restricted, but this restriction was removed in [5]. In [2], Dempwolff provided the infinite analogue of these earlier constructions. In 1986, Grundhöfer [4] gave a synthetic construction for all of these cases which we will outline below. Using this synthetic approach, a family of ovals, called the *Ovali di Roma*, can easily be determined in the Figueroa planes of odd order [this appellation is due to the fact that these ovals were discovered while the author was visiting the Università di Roma—"La Sapienza" under a C.N.R. Visiting Professorship].

2. CONSTRUCTION OF THE FIGUEROA PLANES

Let α be a planar collineation of order 3 of a pappian projective plane $\mathcal{P} = (\mathbf{P}, \mathbf{L}, I)$. The points and lines of $\mathcal{P}$ fall into three disjoint classes each, depending upon the orbit structure under the group $\langle\alpha\rangle$, namely

$$\mathbf{P}_1 = \{p \in \mathbf{P} \mid p^\alpha = p\},$$

$$\mathbf{P}_2 = \{p \in \mathbf{P} - \mathbf{P}_1 \mid p, p^\alpha, p^{\alpha^2} \text{ are collinear}\},$$

$$\mathbf{P}_3 = \{p \in \mathbf{P} \mid p, p^\alpha, p^{\alpha^2} \text{ form a triangle}\}.$$

The line classes $\mathbf{L}_1$, $\mathbf{L}_2$, and $\mathbf{L}_3$ are defined dually. Note that the points of class $\mathbf{P}_2$ are precisely the non-fixed points on a fixed line (a line of class $\mathbf{L}_1$), and so these lines contain no points of class $\mathbf{P}_3$. We define an involutory bijection $\mu : \mathbf{P}_3 \leftrightarrow \mathbf{L}_3$ by $p^\mu = p^\alpha p^{\alpha^2}$ and $\ell^\mu = \ell^\alpha \cap \ell^{\alpha^2}$ for $p \in \mathbf{P}_3$ and $\ell \in \mathbf{L}_3$.

Let $I_{33} = I \cap (\mathbf{P}_3 \times \mathbf{L}_3)$. To construct the Figueroa planes, we modify the incidences in this portion of the incidence relation; in particular we define a new incidence relation $I_\alpha = (I \backslash I_{33}) \cup I^*$ where $I^* \subseteq \mathbf{P}_3 \times \mathbf{L}_3$ and is defined by $pI^*\ell$ if and only if $\ell^\mu I p^\mu$.

The incidence structure $\mathcal{P}_\alpha = (\mathbf{P}, \mathbf{L}, I_\alpha)$ is a projective plane, and if $\mathcal{P}$ had order greater than 8, then $\mathcal{P}_\alpha$ is non-Desarguesian (see [4]).

3. OVALI DI ROMA

In a Desarguesian plane $\mathcal{P}$ of odd order, let Ω be a conic which is invariant under α (i.e., $\Omega^\alpha = \Omega$). Observe that such a conic can contain no $\mathbf{P}_2$ point and has no $\mathbf{L}_2$ tangent. Let π be the subplane fixed by α, and let $\Omega_0 = \Omega \cap \pi$. If Ω_0 is not empty, then it is a conic in π.

Now define, $\Omega^* = \Omega_0 \cup \{\ell^\mu \mid \ell \text{ is a tangent to } \Omega \text{ of class } \mathbf{L}_3\}$.

THEOREM. *The point set Ω^* is an oval in the Figueroa plane $\mathcal{P}_\alpha$.*

PROOF: Since Ω^* contains no $\mathbf{P}_2$ points, any $\mathbf{L}_1$ line can intersect it only in points of Ω_0, and so $\mathbf{L}_1$ lines can meet Ω^* in at most two points. Any $\mathbf{L}_3$ line of $\mathcal{P}_\alpha$ meets Ω^* only at $\mathbf{P}_3$ points. Suppose that an $\mathbf{L}_3$ line m meets Ω^* at three points; then in the Desarguesian plane $\mathcal{P}$, the lines corresponding to these three points must be concurrent at m^μ. This is impossible since these lines are tangents to Ω, and only two tangents can pass through a point if the order of the plane is not even. Thus, we need only consider the intersections of $\mathbf{L}_2$ lines with Ω^*, and since these incidences are the same in both planes, we may work in the Desarguesian plane $\mathcal{P}$. The following lemma will be useful.

LEMMA. *Given a conic in a Desarguesian plane, if two triangles of tangent lines to this conic are perspective from a point, then the corresponding inscribed triangles are perspective from the same point.*

PROOF (OF LEMMA): Let Ω be a conic in a Desarguesian plane and suppose that P_1, P_2, P_3 and Q_1, Q_2, Q_3 are the respective vertices of two triangles each of whose sides is a tangent to Ω and such that the lines P_iQ_i $(i = 1, 2, 3)$ are concurrent at the point S. Label the point of contact of the tangent line P_iP_j (respectively, Q_iQ_j) by p_k (respectively, q_k) so that $\{i, j, k\} = \{1, 2, 3\}$. With this labelling, we wish to show that the lines p_iq_i $(i = 1, 2, 3)$ are concurrent at the point S. Note that the polar with respect to Ω of the point P_i (respectively, Q_i) is the line p_jp_k (respectively, q_jq_k) where $\{i, j, k\} = \{1, 2, 3\}$. Also note that the points $R_i = P_jP_k \cap Q_jQ_k$ (respectively, $r_i = p_jp_k \cap q_jq_k$) where $\{i, j, k\} = \{1, 2, 3\}$ are the poles of the lines p_iq_i (respectively, P_iQ_i) for $i = 1, 2, 3$. By Desargues theorem, the points R_1, R_2, and R_3 are collinear and so their polars p_iq_i $(i = 1, 2, 3)$ are concurrent at a point, say S'. The line determined by the R_i's is the polar of S'. On the other hand, the lines P_iQ_i $(i = 1, 2, 3)$ are concurrent at S, so their poles, the r_i's, are collinear on the polar of S. Since $S' = p_iq_i \cap p_jq_j$, and these are points of the conic, the other two diagonal points of the quadrangle p_i, p_j, q_i, q_j are on the polar of S', in particular, $r_k = p_iq_j \cap p_iq_j$ is on the polar of S' for all k. Thus the polar of S' is the polar of S, and so, $S = S'$. Q.E.D.

Returning now to the proof of the theorem, suppose that points P and Q are $\mathbf{P}_3$ points of Ω^* which are incident with an $\mathbf{L}_2$ line m. Let $R = m \cap \pi$; then $P, P^\alpha, P^{\alpha^2}$ and $Q, Q^\alpha, Q^{\alpha^2}$ are the vertices of two triangles whose sides are tangent to Ω and which are perspective from R. By the lemma, R lies on three secants of Ω and so cannot be a point of Ω_0. Thus, an $\mathbf{L}_2$ line passing through a point of Ω_0 can pass through at most one other point of Ω^*. Now suppose that S was a third $\mathbf{P}_3$ point on m, by two applications of the lemma, if p, q, and s are the points of contact of the tangents $P^\alpha P^{\alpha^2}$, $Q^\alpha Q^{\alpha^2}$, and $S^\alpha S^{\alpha^2}$, respectively, then p, q, and R are collinear as well as p, s, and R; thus p, q, and s would be collinear points of Ω. This contradiction shows that an $\mathbf{L}_2$ line can meet Ω^* *in at most two points*, and so Ω^* is an oval in $\mathcal{P}_\alpha$. Q.E.D.

REMARK. In general, the points of Ω^* do not form an oval in $\mathcal{P}$; for example, in $PG(2, 27)$ the points of Ω^* corresponding to the conic $x^2 + y^2 + z^2 = 0$ lie on the quartic curve $x^2z^2 + y^2z^2 + x^2y^2 = 0$.

REMARK. The above proof does not use finiteness in any essential manner and so shows that the constructed set is an arc in an infinite Figueroa plane. However, whether or not these sets are ovals is still an open question for the infinite case.

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Department of Mathematics
University of Colorado at Denver
1200 Larimer Street
Denver, Colorado 80204 USA
BITNET: WCHEROWITZO@CUDENVER

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