Eigenvectors & Eigenvalues

- **Eigenvector**: A nonzero vector $\vec{x}$ is an eigenvector of an $n \times n$ matrix $A$ if $A\vec{x} = \lambda \vec{x}$ for some scalar $\lambda$.
- **Eigenvalue**: A scalar $\lambda$ is an eigenvalue of an $n \times n$ matrix $A$ if there exists a non-zero vector $\vec{x}$ such that $A\vec{x} = \lambda \vec{x}$, i.e. a nontrivial solution to the homogeneous equation $(A - \lambda I)\vec{x} = \vec{0}$. Such non-zero $\vec{x}$ is called an eigenvector corresponding to eigenvalue $\lambda$.
- **Eigenspace**: The set of all eigenvectors corresponding to an eigenvalue $\lambda_k$ is a subspace of $\mathbb{R}^n$, denoted as $E_{\lambda_k}$. Hence the eigenspace, $E_{\lambda_k} = \text{Null}(A - \lambda_k I)$.
- The eigenspaces of any triangular matrix are the entries of its main diagonal.
- If 0 is not an eigenvalue of $A \iff A$ is invertible (last IMT statement).
- Any set of eigenvectors $\{\vec{x}_1, \vec{x}_2, \ldots, \vec{x}_p\}$ corresponding to distinct eigenvalues $\lambda_1, \lambda_2, \ldots, \lambda_p$ are linearly independent.
- $\lambda$ is an eigenvalue of $A \iff \lambda$ is an eigenvalue of $A^T$.

The Characteristic Equation

- **Characteristic Equation**: The $n^{th}$-degree polynomial in the variable $\lambda$ whose roots are the eigenvalues of an $n \times n$ matrix $A$, $\lambda_1, \lambda_2, \ldots, \lambda_n$.
  
  $$f_A(\lambda) = \det(A - \lambda I) = (\lambda - \lambda_1)(\lambda - \lambda_2)\cdots(\lambda - \lambda_n).$$
- As $f_A(\lambda)$ has at most $n$ roots $\implies A$ has at most $n$ eigenvalues. (For the complex field, exactly $n$ eigenvalues).
- **Algebraic Multiplicity (AM)**: An eigenvalue $\lambda_0$ that occurs $k$ times as a root of $f_A(\lambda)$ has algebraic multiplicity $k$. Hence, $f_A(\lambda) = (\lambda - \lambda_0)^k g(\lambda)$.
- **Geometric Multiplicity (GM)**: The geometric multiplicity of an eigenvalue $\lambda_i$ is the dimension of its eigenspace, $GM_i = \dim(\text{Null}(A - \lambda_i I)) = \dim(E_{\lambda_i})$. Just count the number of vectors.
- Two $n \times n$ matrices $A$ and $B$ are **similar** if there exists an invertible matrix $P$ such that: $A = PBP^{-1}$.
- Similar matrices have the same characteristic equation and thus, the same eigenvalues. Row equivalent matrices do not (generally) have the same eigenvalues.

Diagonalization

- **Diagonal Matrix**: An $n \times n$ matrix $D$ is diagonal if all entries not on the main diagonal are equal to zero. Sometimes we denote $D = \text{diag} \{\lambda_1, \lambda_2, \ldots, \lambda_n\}$ where $\lambda_i$ are the main diagonal entries.
- **Diagonalizable**: An $n \times n$ matrix $A$ is said to be diagonalizable if $A$ is similar to a diagonal matrix $D$, i.e. there exists invertible matrix $P$ such that $A = PDP^{-1}$, or $P^{-1}AP$ is a diagonal matrix.
- **Eigenspace**: A linearly independent set of eigenvectors of $A$ that span $\mathbb{R}^n$ is called an eigenspace of $A$.
- An $n \times n$ matrix $A$ is diagonalizable $\iff A$ has an eigenspace $\iff$ AM=GM for all eigenvalues of $A$.
- **Method for Diagonalizing a Matrix $A_{n \times n}$**:
  1. Find the eigenvalues of $A$. Suppose you have $p$ eigenvalues.
  2. Find a basis for the eigenspace $E_{\lambda_i}$ for each eigenvalue $\lambda_i$ of $A$.
  3. If you have enough independent vectors, i.e. if $\sum_{i=1}^{p} GM_i = \sum_{i=1}^{p} \dim(E_{\lambda_i}) = n$, then concatenate the sets of basis vectors for each eigenspace to form an eigenspace of $A$. This set of $n$ vectors forms the columns of $P = [\vec{x}_1 \quad \vec{x}_2 \quad \cdots \quad \vec{x}_n]$.
  4. Set $D = \text{diag} \{\lambda_1, \lambda_2, \ldots, \lambda_n\}$ such that $A\vec{x}_i = \lambda_i \vec{x}_i$ for $i = 1, 2, \ldots, n$.
  5. Check that: $AP = PD$ (Optional).
- Diagonalization can also be considered change of bases for the linear operator, $T$, corresponding to the matrix.
- If diagonalizable, $A = PDP^{-1}$ for some invertible $P$ and $A^k = PD^kP^{-1}$.