This document will contain homework and notes from Math 5070 Fall 2012.

Aug 21 inf, sup, limit of a sequence

Aug 23 Bolzano-Weierstrass theorem, Cauchy condition, lim inf and lim sup of a bounded sequence. Assignment: Read all of sections 1 – 19 to review the prerequisites. Homework 1 due Sep 4: Exercises 5.7, 9.4, 20.1, 20.9.

Aug 28 lim inf and lim sup in general case, root and ratio inequality.

Aug 30 Proofs of the equivalent statements for lim sup and lim inf. 22-24: Series, series with nonnegative terms, $2^n$ test. Homework 2 Due Sep 11: 20.8, 25.3, 27.2, 28.4

Sep 4 Absolute convergence of series, power series

Sep 6 Non-absolute convergence, Dirichlet test, Abel test

Sep 11 Metric spaces, examples: $\ell^1, \ell^\infty$, convergent sequences. Homework 3 Due Sep 18: 1. 35.4, 2. 35.7, 3. 36.11, 4. Define $\mathbb{R}^\infty$ to be the space of all real sequences equipped with the distance function

$$d(x, y) = \sum_{n=1}^{\infty} \frac{1}{2^n} \frac{|x_n - y_n|}{1 + |x_n - y_n|}$$

Show that $(\mathbb{R}^\infty, d)$ is a metric space, and $\lim_{k \to \infty} x^{(k)} = L$ in $(\mathbb{R}^\infty, d)$ if and only if $\lim_{k \to \infty} x_n^{(k)} = L_n$ for all $n$.

Sep 13 $\ell^2$, inner product spaces, closure, closed sets.

Sep 18 Solution of Homework 3, duality between $\ell^1$ and $\ell^\infty$. Homework 4, due September 25: 38.5, 38.7, 38.8, 38.13.

Sep 20 Open sets

Sep 25 40: Continuous functions on metric spaces, 41: relative metric Homework 5 due October 2: 39.9, 39.10, 40.10, 40.17

Sep 27 Relative metric, compact space

Oct 2 Characterization of compact metric spaces as sequentially compact. Homework 6 due October 9: 41.4, 42.4 (do not use the methods of 43), 43.7, 44.2

Oct 4 Compact sets and continuous functions, definition of uniform continuity.

Oct 9 Complete metric spaces, relation with closed, compact. Homework 7 due October 16: 44.5, 46.1, 46.2, 46.10

Oct 11 Hint for 46.10, properties of compact sets: problem 42.5, compact $\iff$ complete and totally bounded, Theorem 44.5, Definition 45.1, Theorem 45.3 (no proof) and generalization to open subsets of $\mathbb{R}^n$.

Oct 16 Banach contraction principle. Review and problem solving. Recommended review problems: 5.1-7, 9.3-9, 15.1-5, 16.7.8, 18.1-5, 19.1-4, 20.1.6-17,19-22, 21.2-6, 25.1-4, 26.1-9, 27.1-5, 28.1-3,6, 35.1-8, 36.2-9, 37.1-10, 38.1,3-11,13,14 39.1-10, 40.2-14, 41.2-4, 42.1-7,10,12, 43.1-4, 44.1-6(c)

Oct 18 Midterm.

Oct 23 Connected metric spaces

Oct 25 Connectivity (conclusion), Riemann-Stieltjes integral. Still no homework, that will be Tuesday.
Oct 30  Riemann-Stieltjes integral. Homework 8 due November 6: 51.7, 51.13, 51.17, 51.20(c)

Nov  1    Homework problems

Nov  6    Uniform convergence. Homework 9 due November 13: 1. Let \( f_n, f : M_1 \setminus \{a\} \to M_2 \) where \((M_1, d_1)\) and \((M_2, d_2)\) are metric spaces, \((M_2, d_2)\) is complete, and \(a \in M_1\). Prove that if \( f_n \Rightarrow f \) on \( M_1 \setminus \{a\} \) and \( \lim_{x \to a} f_n(x) = b_n \) exists for all \( n \), then \( \lim_{n \to \infty} b_n = b \) exists, and \( \lim_{x \to a} f(x) = b \). That is, \( \lim_{n \to \infty} \lim_{x \to a} f_n(x) = \lim_{x \to a} \lim_{n \to \infty} f_n(x) \). Provide an example where \( f_n \to f \) on \( M_1 \setminus \{a\} \), but \( \lim_{n \to \infty} \lim_{x \to a} f_n(x) \neq \lim_{x \to a} \lim_{n \to \infty} f_n(x) \) (and all limits exist). 2. 60.2 3. 60.3 4. 60.9

Nov  8    Uniform Cauchy condition, spaces \( C[a, b] \) and \( C(K) \), uniform convergence under integral sign

Nov 13   Counterexample to switching limits under integral sign, uniform convergence and differentiation. Homework 10 due November 27: 1. 60.10, 2. 62.1, 3. 62.2, 4. 62.4

Nov 15   Series of functions, M-test, uniform convergence of power series, Abel’s theorem (no proof yet)

Nov 28   Abel’s theorems, equicontinuous functions. Homework 11 due December 4: 1. 64.2, 2. Prove that if \( \{f_n\} \) are continuous and uniformly bounded on \([a, b]\), \( g \) is continuous on \([a, b] \times [a, b]\), and \( h_n(x) = \int_a^b g(x, y)f_n(y)dy \), then there exists a subsequence \( \{h_{n_k}\} \) uniformly convergent on \([a, b]\). (2 problems 8 points each)

Nov 30   Proof of Arzéla-Ascoli theorem, Banach spaces, space \( C(K) \)

Dec  4   Inverse function theorem

Dec  6   Implicit function theorem, examples

Dec 11   Final 5-7pm.