1. 5.7 Let $X$ and $Y$ be sets of real numbers with least upper bounds $a$ and $b$, respectively. Prove that $a + b$ is the least upper bound of the set $X + Y = \{x + y | x \in X, y \in Y\}$.

Solution. Because $a = \sup X$, $a$ is upper bound on $X$, so $\forall x \in X : x \leq a$. Likewise, $\forall y \in Y : y \leq b$. Let $z \in X + Y$. Then there exist $x \in X, y \in Y$ such that $z = x + y$. Then $z = x + y \leq a + b$ so $a + b$ is upper bound on $X + Y$.

Suppose $c < a + b$ is also upper bound on $X + Y$. Then $c = a + b - \varepsilon, \varepsilon > 0$. Since $a - \varepsilon/2 < a = \sup X$, there exists $x \in X$ such that $x > a - \varepsilon/2$ and likewise there exists $y \in Y$ such that $y > b - \varepsilon/2$. Then $x + y > a - \varepsilon/2 + b - \varepsilon/2 = a + b - \varepsilon = c$, so $c$ is not an upper bound on $X + Y$.

2. 9.4 Let $A$ and $B$ be sets such that $A \subset B$. Prove that if $A$ is uncountable, then $B$ is also uncountable.

Solution: Suppose $B$ is countable. By Corollary 9.2, $A$ is countable, which is a contradiction.

3. 20.1 Prove that $\liminf_{n \to \infty} (-1)^n = -1$.

Solution. Since $a_n = (-1)^n \geq -1$, $\liminf_{n \to \infty} (-1)^n \leq -1$. Subsequence $a_{2k+1} = -1$ has limit $-1$, hence $\liminf_{n \to \infty} (-1)^n \leq -1$.

In other words: $-1$ is a lower bound on $L_a$, the set of all subsequential limits, and $-1 \in L_a$. Hence $\liminf_{n \to \infty} (-1) = \inf L_a = -1$.

4. 20.9 Let $\{a_n\}$ be a bounded sequence such that every convergent subsequence has limit $L$. Prove that $\lim_{n \to \infty} a_n = L$.

Solution. Since $\{a_n\}$ is bounded, at least one convergent subsequence exists, so $L_a \neq \emptyset$. It is given that $x \in L_a$ implies $x = L$. Hence $L_a = \{L\}$ and $\liminf_{n \to \infty} a_n = \inf L_a = L$ and $\limsup_{n \to \infty} a_n = \sup L_a = L$, hence $\lim_{n \to \infty} a_n = L$. 

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