1. 60.10 Prove that $C[a, b]$ is a complete metric space with the metric $d(f, g) = \sup \{|f(x) - g(x)| : x \in [a, b]\}$.

Proof. Suppose $\{f_n\}$ is Cauchy in $C[a, b]$. Then $\{f_n\}$ is uniformly Cauchy and thus converges uniformly to some $f$. Since $f_n$ are continuous, $f$ is continuous.

2. 62.1 Prove that the series $\sum_{n=1}^{\infty} \frac{1}{n+x}$ converges uniformly on $[0, \infty)$.

Proof. We have $\left|\frac{1}{n+x}\right| \leq \frac{1}{n^2}$ for all $x \in [0, \infty)$ and $\sum_{n=1}^{\infty} \frac{1}{n^2}$ converges, so uniform convergence follows from the $M$-test.

3. 62.(a) Prove that the series $\sum_{n=1}^{\infty} x^n (1 - x)$ converges pointwise but not uniformly on $[0, 1]$. (b) Prove that $\sum_{n=1}^{\infty} (-1)^n x^n (1 - x)$ converges uniformly on $[0, 1]$.

Solution. (a) For any $x \in [0, 1)$, $\sum_{n=1}^{\infty} x^n (1 - x)$ converges as geometric series with ratio $x$. For $x = 1$, this is a series of all zeros, thus convergent. But

$$T_n(x) = \sum_{k=n+1}^{\infty} x^k (1 - x) = (1 - x) x^{n+1} \frac{1}{1 - x} = x^{n+2}$$

and uniform convergence of the series is the same as $T_n(x) \to 0$. But

$$\sup_{x \in [a, b]} |T_n(x)| \geq T_n \left(1 - \frac{1}{n + 2}\right) \to \frac{1}{e} > 0.$$

(b) Again, for $x \in [0, 1)$, $\sum_{n=1}^{\infty} (-1)^n x^n (1 - x)$ converges as geometric series with ratio $-x$. For $x = 1$, this is a series of all zeros. Now

$$T_n(x) = \sum_{k=n+1}^{\infty} (-1)^k x^k (1 - x) = (-1)^{n+1} (1 - x) x^{n+1} \frac{1}{1 + x} = \frac{(1 - x) x^{n+2}}{1 + x}$$

Thus

$$|T_n(x)| \leq f(x) = (1 - x) x^{n+2}$$

where $f(0) = f(1) = 0$ and $f'(x) = (-1) x^{n+2} + (1 - x) (n + 2) x^{n+1} = x^{n+1} ((n + 2) - (n + 3) x) = 0$ for $x \in (0, 1)$ iff $x = \frac{n+2}{n+3} = 1 - \frac{1}{n+3}$, so

$$\sup_{x \in [0, 1]} |T(x)| \leq f \left(1 - \frac{1}{n + 3}\right) = \frac{1}{n + 3} \left(1 - \frac{1}{n + 3}\right)^{n+2} \to 0 \cdot \frac{1}{e} = 0 \text{ as } n \to \infty.$$

4. 62.4 Prove Dini’s theorem for series: Let $\{u_k\}$ be a sequence of nonnegative continuous functions on a compact metric space $M$. If $\sum_{n=1}^{\infty} u_n$ converges pointwise to a continuous function $f$ on $M$, then $\sum_{n=1}^{\infty} u_n$ converges uniformly to $f$ on $M$.

Solution. Let $g_n = f - \sum_{k=1}^{n} u_k$. Then $0 \leq g_{n+1} \leq g_n$, $g_n \to 0$ pointwise, and since $f$ and $u_k$ are continuous, $g_n$ are continuous. By Dini’s theorem, $g_n \to 0$ on $M$, which is the same as the series $\sum_{n=1}^{\infty} u_n$ converges uniformly to $f$ on $M$. 

Math 5070 Fall 2012 Homework 10 solutions