1. 38.5 Let $X$ be subset of a metric space $M$. We say that a point $x$ in $M$ is an accumulation point of $X$ there exists a sequence $\{x_n\}$ in $X$ such that $\lim_{n \to \infty} x_n = x$ and $x_n \neq x$ for all $n$. Denote by $X^a$ the set of all accumulation points of $X$.

(a) Prove that $X$ is closed if and only if $X^a \subset X$.
(b) Prove that if $X$ is a bounded infinite subset of $\mathbb{R}$, then $X^a \neq \emptyset$.
(c) Prove that if $X$ is an uncountable subset of $\mathbb{R}$, then $X^a \neq \emptyset$.

Solution.

(a) We have defined that $X$ is closed if $\lim_{n \to \infty} x_n \in X$ for every sequence $\{x_n\}$ in $X$ which converges in $M$.

$\iff$: Let $x$ be an accumulation point of $X$. Then there exists a sequence $\{x_n\}$ in $X$ such that $\lim_{n \to \infty} x_n = x$ in $M$, and because $X$ is closed, $x \in X$.

$\iff$: Suppose that $X^a \subset X$, $\{x_n\}$ is a sequence in $X$, and $\lim_{n \to \infty} x_n = x$. We need to show that $x \in X$. If $x_n = x$ for some $n$, then $x_n \in X$ because $x \in X$. If $x_n \neq x$ for all $n$, then $x \in X^a \subset X$.

(b) Because $X$ is infinite, we can choose a sequence $\{x_n\}$ in $X$ with all $x_n$ different. Because $X$ is bounded, the sequence $\{x_n\}$ is bounded, and by the Weierstrass theorem, there is a convergent subsequence $x_{n_k} \to x$. We will show that $x \in X^a$. If $x_{n_k} \neq x$ for all $k$, we are done. Otherwise, since all $x_{n_k}$ are different, $x_{n_k} = x$ for only one $k$, and by removing this term from the subsequence we get a subsequence which of numbers from $X$ all different from $x$, which also converges to $x$.

Another solution. Because $X$ is bounded, $X \subset (a_0, b_0)$ for some real $a_0, b_0$. Then $X \cap (a_0, b_0)$ is infinite. If we already have $a_k, b_k$ such that $X \cap (a_k, b_k)$ is infinite, at least one of the sets

$$X \cap \left( a_k, \frac{a_k + b_k}{2} \right), \quad X \cap \left( \frac{a_k + b_k}{2}, b_k \right)$$

is infinite. Choose $(a_{k+1}, b_{k+1})$ to be one of the intervals $(a_k, \frac{a_k + b_k}{2})$ or $(\frac{a_k + b_k}{2}, b_k)$ such that $X \cap (a_k, b_k)$ is infinite. By induction, we get intervals

$$(a_0, b_0) \supset (a_1, b_1) \supset \cdots \supset (a_k, b_k) \supset \cdots$$

of lengths

$$b_k - a_k = \frac{b_0 - a_0}{2^k} \to 0, \quad k \to \infty.$$ 

Since $a_k$ is increasing and bounded (by $b_0$), it has a limit $L$. Because $\lim_{k \to \infty} b_k - a_k = 0$, $b_k$ has the same limit $L$. Since the set $X \cap (a_k, b_k)$ has at least two elements (in fact, it is infinite), we can choose $x_k \in X \cap (a_k, b_k)$, $x_k \neq L$, and by the squeeze theorem, $\lim_{k \to \infty} x_k = L$.

(c) We first show that if $x \in X$ but $x \notin X^a$ then

$$\exists \varepsilon_x > 0 : (x - \varepsilon_x, x + \varepsilon_x) \cap X \neq \{x\}. \quad (1)$$
Otherwise
\[ \forall \varepsilon > 0 : (x - \varepsilon, x + \varepsilon) \cap X \neq \{x\} , \]

Because \( x \in (x - \varepsilon, x + \varepsilon) \cap X \) always and choosing \( \varepsilon = 1/n \) in turn we get
\[ \forall n \in \mathbb{N} \exists x_n \in (x - 1/n, x + 1/n) \cap X, x_n \neq x. \]

Since \( x_n \to x, n \to \infty \), and \( x_n \neq x \) for all \( n \), it follows that \( x \in X^a \).

Taking the \( \varepsilon_x \) from (1), we have
\[ X \subset \bigcup_{x \in X} (x - \varepsilon_x/2, x + \varepsilon_x/2) \]

and the sets \( (x - \varepsilon_x/2, x + \varepsilon_x/2) \) are disjoint; if
\[ (x - \varepsilon_x/2, x + \varepsilon_x/2) \cap (y - \varepsilon_y/2, y + \varepsilon_y/2) \neq \emptyset \]

for some \( x, y \in X \), \( x \neq y \), then \( |x - y| < \varepsilon_x/2 + \varepsilon_y/2 \leq \max \{ \varepsilon_x, \varepsilon_y \} \) and so either \( y \in (x - \varepsilon_x, x + \varepsilon_x) \cap X \) or \( x \in (y - \varepsilon_y, y + \varepsilon_y) \cap X \), which is a contradiction. By the density of rationals, there exists a rational number \( r_x \in (x - \varepsilon_x/2, x + \varepsilon_x/2) \) and since the intervals are disjoint, \( r_x \) is different for different \( x \), and \( x \mapsto r_x \) is a one to one mapping between \( X \) and a subset of rationals, which is countable.

Alternative solution. Define \( X_n = X \cap [n, n + 1] \). If all \( X_n \) were finite, then
\[ X = \bigcup_{n=-\infty}^{\infty} X_n \]

is countable as the countable union of finite sets. Thus there exists \( X_n \) which is infinite. Because \( X_n \) is bounded, \( \emptyset \neq X_n^a \) and because \( X_n^a \subset X^a \), it holds that \( X^a \neq \emptyset \).

2. 38.7 Let \( \delta^{(k)} \) be the sequences
\[ \delta^{(1)} = (1, 0, 0, \ldots) \]
\[ \delta^{(2)} = (0, 1, 0, \ldots) \]
\[ \vdots \]

that is, for each \( k \), \( \delta^{(k)} \) is the sequence \( \{\delta^{(k)}_n\}_{n=1}^{\infty} \) defined by \( \delta^{(k)}_k = 1, \delta^{(k)}_n = 0 \) if \( n \neq k \). Prove that the set \( X = \{\delta^{(k)}|k \in \mathbb{N}\} \) is closed subset of \( \ell^1, \ell^2, c^0 \), and \( \ell^\infty \).

Solution. First, \( X \) is a subset of all those spaces:
\[ \sum_{n=1}^{\infty} |\delta^{(k)}_n| = 1 \implies \delta^{(k)} \in \ell^1 \]
\[ \sum_{n=1}^{\infty} |\delta^{(k)}_n|^2 = 1 \implies \delta^{(k)} \in \ell^2 \]
\[ \forall n \in \mathbb{N} : |\delta^{(k)}_n| \leq 1 \implies \delta^{(k)} \in \ell^\infty \]
\[ \lim_{n \to \infty} \delta^{(k)}_n = 0 \implies \delta^{(k)} \in c^0 \]

2
where the last statement follows from the fact that the sequence \( \delta_n^{(k)} = 0 \) for all \( n > k \).

Now, if \( i \neq j \), then
\[
\begin{align*}
    d_1 (\delta^{(i)}, \delta^{(j)}) &= 1 + 1 = 2 \\
    d_2 (\delta^{(i)}, \delta^{(j)}) &= \sqrt{1+1} = \sqrt{2} \\
    d_\infty (\delta^{(i)}, \delta^{(j)}) &= 1
\end{align*}
\]

Since always \( d(\delta^{(i)}, \delta^{(j)}) \geq 1 \) for \( i \neq j \), the only sequences from \( X \) that are convergent are those that are eventually constant, and so \( X \) has no accumulation points. The result follows from problem 1(a).

3. 38.8 Let \( X \) and \( Y \) be closed subsets of \( \mathbb{R} \). Prove that \( X \times Y \) is a closed subset of \( \mathbb{R}^2 \). State and prove a generalization to \( \mathbb{R}^n \).

Solution. Consider \( X_1, \ldots, X_n \subset \mathbb{R} \) closed. The metric in \( \mathbb{R}^n \) is given by
\[
d((x_1, \ldots, x_n), (y_1, \ldots, y_n)) = \sqrt{\sum_{i=1}^{n} |x_i - y_i|^2}.
\]

Suppose \( \{x^{(k)}\} \) is a sequence in \( X_1 \times \cdots \times X_n \) such that \( \lim_{k \to \infty} x^{(k)} = x \) in \( \mathbb{R}^n \), that is
\[
\lim_{k \to \infty} d\left(x^{(k)}, x\right) = 0.
\]

Then for all \( i = 1, \ldots, n \),
\[
|x_i^{(k)} - x_i| \leq d\left(x^{(k)}, x\right) \to 0 \text{ as } k \to \infty,
\]
thus \( \lim_{k \to \infty} x_i^{(k)} = x_i \) in \( \mathbb{R} \). Since \( X_i \) are closed, \( x_i \in X_i \) for all \( i = 1, \ldots, n \), and, consequently,
\[
x = (x_1, \ldots, x_n) \in X_1 \times \cdots \times X_n.
\]

4. 38.13 Let \( M \) be a metric space. Prove the following:
(a) \( \overline{X} = \overline{\overline{X}} \) for all \( X \subset M \)
(b) \( \overline{X} \) is closed for all \( X \subset M \)
(c) For all \( X, Y \subset M \), if \( X \subset Y \subset M \), then \( \overline{X} \subset \overline{Y} \)
(d) \( \overline{X \cup Y} = \overline{X} \cup \overline{Y} \) for all \( X, Y \subset M \)
(e) If \( Y \) is a closed subset of \( M \) such that \( X \subset Y \), then \( \overline{X} \subset \overline{Y} \)
(f) If \( X \subset M \), then \( \overline{X} = \bigcap \{Y | X \subset Y \subset M, Y \text{ is closed}\} \)

Solution.
(a) Since any set is a subset of its closure, we have \( \overline{X} \subset \overline{\overline{X}} \). We need to prove \( \overline{\overline{X}} \subset \overline{X} \). Let \( x \in \overline{\overline{X}} \). Then there exists a sequence \( \{x_n\} \subset \overline{X} \), \( x_n \to x \) as \( n \to \infty \). Let \( m \in \mathbb{N} \). Since \( x_n \to x \) as \( n \to \infty \), there exists some \( n \) such that \( d(x_n, x) < 1/2m \). Since \( x_n \in \overline{X} \), there exists a sequence \( \{y_k\} \subset X \), \( y_k \to x_n \) as
\[ k \to \infty, \text{ and so there exists } k \text{ such that } d(y_k, x_n) < 1/2m. \text{Put } z_m = y_k. \text{ Then } z_m \in X \text{ and} \]

\[ d(z_m, x) \leq d(z_m, x_n) + d(x_n, x) < \frac{1}{2m} + \frac{1}{2m} = \frac{1}{m}. \]

We have constructed a sequence \( \{z_m\} \subset X, z_m \to x \text{ as } m \to \infty, \text{ so } x \in \overline{X}. \)

(b) Since \( \overline{X} = \overline{X} \) by (a), \( \overline{X} \) is closed.

(c) If \( x \in \overline{X}, \text{ then } x = \lim_{n \to \infty} x_n \text{ for some } \{x_n\} \subset X. \text{ Since } X \subset Y, \text{ also } \{x_n\} \subset Y, \text{ so } x \in \overline{Y}. \)

(d) Since \( X \subset X \cup Y \) and \( Y \subset X \cup Y, \) we have by (c) \( \overline{X} \subset \overline{X \cup Y} \) \( \text{and } \overline{Y} \subset \overline{X \cup Y}, \) \( \text{so } \overline{X} \cup \overline{Y} \subset \overline{X \cup Y}. \) \text{ For the opposite inclusion, let } x \in \overline{X \cup Y}. \text{ Then } x = \lim_{n \to \infty} x_n, \text{ where all } x_n \in X \cup Y. \text{ Either infinitely many } x_n \in X, \text{ in which case } x \in \overline{X} \text{ because the subsequence of } x_n \in X \text{ converges to } x, \text{ or infinitely many } x_n \in Y \text{ and then } x \in \overline{Y} \text{ (or both).} \)

(e) Since \( Y \) is closed, \( Y = \overline{Y}. \) \text{ By (c), } X \subset Y \text{ gives } \overline{X} \subset \overline{Y} = Y. \)

(f) Denote \( \mathcal{F} = \{Y | X \subset Y \subset M, \ Y \text{ is closed}\} \text{, the family of all closed sets in } M \text{ which contain } X. \text{ Since all sets in } \mathcal{F} \text{ are closed, and the intersection of a family of closed sets is closed, } \bigcap \mathcal{F} \text{ is closed. Since } X \text{ is subset of any set } Y \text{ in } \mathcal{F}, \text{ it is contained in their intersection, so } X \subset \bigcap \mathcal{F}. \text{ By (e), } \overline{X} \subset \bigcap \mathcal{F}. \text{ But } X \subset \overline{X} \text{ and } \overline{X} \text{ is closed by (b), so } \overline{X} \in \mathcal{F}, \text{ hence } \bigcap \mathcal{F} \subset \overline{X}. \)