1 General definition of sup and lim sup

The concept of supremum applies to extended reals also: $s \in \mathbb{R}^*$ is defined to be a supremum of $A \subset \mathbb{R}^* = \mathbb{R} \cup \{-\infty, \infty\}$ if

$$\forall x \in A : x \leq s$$
$$\forall b < s \ \exists x \in A : x > b$$

Exercise: Show that supremum is unique so we are entitled to write $s = \sup A$.

Extend the definition of limit to extended reals: $L = \lim_{n \to \infty} a_n$ if

$$\forall a > L \ \exists N_1 \in \mathbb{N} \ \forall n > N_1 : a_n < a$$
$$\forall b < L \ \exists N_2 \in \mathbb{N} \ \forall n > N_2 : a_n > b$$

We have equivalent definitions of $\limsup_{n \to \infty} a_n$:

1. $\limsup_{n \to \infty} a_n = \sup \{\lim_{k \to \infty} a_{n_k} | \{a_{n_k}\} \text{ is a subsequence of } \{a_n\} \text{ that has a limit}\}$
2. $L$ is $\limsup_{n \to \infty} a_n$ if
   $$\forall a > L : \{n \in \mathbb{N} | a_n \geq a \} \text{ is finite}$$
   $$\forall b < L : \{n \in \mathbb{N} | a_n > b \} \text{ is infinite}$$
3. $\limsup_{n \to \infty} a_n = \lim A_n$, $A_n = \sup \{a_n, a_{n+1}, \ldots\}$

For bounded sequences, we have proved that $(3) \Rightarrow (1)$ in class, and we know that the quantity defined by $(3)$ always exists and the quantity defined by $(1)$ is unique. So the $\limsup$ as defined by $(3)$ and $(1)$ are the same. Theorem 20.3 in the book shows that $(1) \Rightarrow (2)$, again for the case of bounded sequences (their $L + \varepsilon = a$ and $L - \varepsilon = b$ here), and the quantity defined by $(1)$ always exists.

Exercise: Show that the implications $(3) \Rightarrow (1)$ and $(1) \Rightarrow (2)$ stay true whether $\{a_n\} \subset \mathbb{R}$ is bounded or not. However, to keep it simpler, we do not allow $\pm \infty$ as the values of $a_n$ here.

2 In-class activity

Group 1 Show that for any sequence in $\mathbb{R}$, bounded or not, there is at most one $L \in \mathbb{R}^*$ that satisfies the conditions in (2):

$$\forall a > L : \{n \in \mathbb{N} | a_n \geq a \} \text{ is finite}$$
$$\forall b < L : \{n \in \mathbb{N} | a_n > b \} \text{ is infinite}$$

Group 2 Let $a_n > 0$ for all $n$. Show that

$$\limsup_{n \to \infty} \sqrt[n]{a_n} \leq \limsup_{n \to \infty} \frac{a_{n+1}}{a_n}.$$  

Hint: show that $\limsup_{n \to \infty} \sqrt[n]{a_n} < a \Rightarrow \limsup_{n \to \infty} \sqrt[n]{a_n} \leq a$, using (2).
Solutions:

**Group 1** Let $L_1 \neq L_2$, and without loss of generality, $L_1 < L_2$. Choose $M \in \mathbb{R}$ such that $L_1 < M < L_2$. From the first part of property (2) for $L = L_1$, the set $\{n \in \mathbb{N}|a_n \geq M\}$ is finite, because $M > L_1$. From the second part of property (2) for $L = L_2$, the set $\{n \in \mathbb{N}|a_n > M\}$ is infinite, because $M < L_2$. Since $\{n \in \mathbb{N}|a_n > M\} \subset \{n \in \mathbb{N}|a_n \geq M\}$, this is a contradiction.

**Group 2** Let 

$$\limsup_{n \to \infty} \frac{a_{n+1}}{a_n} < a,$$

for some $a$. We will show that 

$$\limsup_{n \to \infty} \sqrt[n]{a_n} \leq a.$$ 

If $a = \infty$, we are done. So, suppose $a < \infty$. Since $\limsup_{n \to \infty} \frac{a_{n+1}}{a_n} < a$, there are only finitely many $n$ such that $\limsup_{n \to \infty} \frac{a_{n+1}}{a_n} \geq a$. Thus, there exists an $N$ such that for all $n \geq N$, 

$$\frac{a_{n+1}}{a_n} < a.$$ 

By induction, we have for all $n \geq N$, 

$$a_n \leq a^{n-N} a_N.$$ 

Taking the $n$-th root, for all $n \geq N$, 

$$\sqrt[n]{a_n} \leq a^{\frac{n-N}{n}} a_N^{1/n}$$

so 

$$\limsup_{n \to \infty} \sqrt[n]{a_n} \leq \limsup_{n \to \infty} a^{\frac{n-N}{n}} a_N^{1/n} = \lim_{n \to \infty} a^{\frac{n-N}{n}} a_N^{1/n} = a^{1} \cdot 1 = a.$$ 

Now, if $\limsup_{n \to \infty} \sqrt[n]{a_n} > \limsup_{n \to \infty} \frac{a_{n+1}}{a_n}$, there exists $a$ such that 

$$\limsup_{n \to \infty} \sqrt[n]{a_n} > a > \limsup_{n \to \infty} \frac{a_{n+1}}{a_n},$$

but from the above, $\limsup_{n \to \infty} \sqrt[n]{a_n} \leq a$, contradiction.