Homework 10: Show that a Riemann integrable function is Lebesgue integrable (the integral for the Lebesgue measure exists), and the values of the two integrals are the same. Hint: Turn sequences of upper and lower sums into sequences of integrals of step functions, and show that the sequences of step functions are Cauchy.

Solution. Riemann integral is defined for bounded functions $f$ on a closed and bounded interval $[a, b]$ as follows: for any partition $P = \{ x_0 = a, x_1, \ldots, x_n = b \}$, the corresponding lower sum $L(f, P)$ and upper sum $U(f, P)$ are defined by

$$L(f, P) = \sum_{i=1}^{n} \inf_{x \in [x_{i-1}, x_i]} f(x) (x_i - x_{i-1})$$

$$U(f, P) = \sum_{i=1}^{n} \sup_{x \in [x_{i-1}, x_i]} f(x) (x_i - x_{i-1})$$

Function $f$ is Riemann integrable if $\sup_P L(f, P) = \inf_P U(f, P)$, and the integral $\int_a^b f \, dx$ then equals to this common value. For every partition $P$, define the functions

$$\phi_{f, P}(x) = \sum_{i=1}^{n} \inf_{x \in [x_{i-1}, x_i]} f(x) \text{ if } x \in (x_{i-1}, x_i)$$

$$\psi_{f, P}(x) = \sum_{i=1}^{n} \sup_{x \in [x_{i-1}, x_i]} f(x) \text{ if } x \in (x_{i-1}, x_i)$$

At the nodes $x_i$, the functions $\phi_{f, P}$ and $\psi_{f, P}$ are 0. Then $\phi_{f, P}$ and $\psi_{f, P}$ are step functions, and by definition, the lower and upper sums are their integrals,

$$L(f, P) = \int \phi_{f, P}, \quad U(f, P) = \int \psi_{f, P},$$

(with respect to Lebesgue measure, here simply interval length), and

$$\phi_{f, P} \leq f \leq \psi_{f, P}$$

except at the nodes $x_i$.

It is known from the theory of Riemann integration (using the definitions of infimum and supremum, and taking common refinement of partitions) that if $f$ is Riemann integrable, then there exists a sequence of partitions $P_k$ such that

$$\int_a^b f \, dx = \lim_{k \to \infty} L(f, P_k) = \lim_{k \to \infty} U(f, P_k).$$

and $P_{k+1}$ is a refinement of $P_k$, thus

$$\phi_{f, P_k} \leq \phi_{f, P_{k+1}} \leq f \leq \psi_{f, P_{k+1}} \leq \psi_{f, P_k}$$

except at the nodes of the partitions $P_k$, which is a countable set. Hence,

$$\int |\phi_{f, P_{k+1}} - \phi_{f, P_k}| = \int \phi_{f, P_{k+1}} - \phi_{f, P_k} = \int \phi_{f, P_{k+1}} - \phi_{f, P_{k}} = L(f, P_{k+1}) - L(f, P_k) = |L(f, P_{k+1}) - L(f, P_k)|$$

Since the sequence $\{L(f, P_k)\}$ converges, it is Cauchy sequence in $\mathbb{R}$, and, consequently, $\{\phi_{f, P_k}\}$ is $L^1$ Cauchy sequence of step maps. Similarly, $\{\psi_{f, P_k}\}$ is $L^1$ Cauchy sequence of step maps. By Lemma 3.1, $\phi_{f, P_k}$ and $\psi_{f, P_k}$ converge a.e. on $[a, b]$, and since $\phi_{f, P} \leq f \leq \psi_{f, P}$ a.e., they converge to $f$ a.e. Thus the limits of the sequences of the integrals of the step maps $\phi_{f, P}$ and $\psi_{f, P}$ equal to the Lebesgue integral of $f$. Since the integrals of the step maps equal to the lower and upper Riemann sums, whose limit is the Riemann integral, the Riemann integral equals to the Lebesgue integral.