Homework 4
Problem 3 p. 172

Let \( \{ f_n \} \) be a sequence of measurable functions. Show that the set of those \( x \) such that \( \{ f_n (x) \} \) converges is a measurable set.

Solution. We have \( f_n : X \to \mathbb{R} \) or \( \mathbb{C} \) where \( X \) is a measurable space. The range is reals or complex numbers, otherwise \( f \) would be called a map. Consider first \( f_n : X \to \mathbb{R} \).

By the definition of \( \limsup \), \( \limsup_{n \to \infty} f_n (x) < a \) is equivalent to
\[
\exists z < a \exists N \forall n \geq N : f_n (x) < z.
\]

Translation into set operations gives
\[
\left\{ x \mid \limsup_{n \to \infty} f_n (x) < a \right\} = \bigcup_{m=1}^{\infty} \bigcup_{N=1}^{\infty} \bigcap_{n=N}^{\infty} \left\{ x \mid f_n (x) < a - \frac{1}{m} \right\},
\]
which implies that the sets
\[
S (a) = \left\{ x \mid \limsup_{n \to \infty} f_n < a \right\}
\]
are measurable for any real \( a \). Since
\[
S (\infty) = \bigcup_{m=1}^{\infty} S (m),
\]
\( S (\infty) \) is also measurable. Similarly,
\[
T (a) = \left\{ x \mid \limsup_{n \to \infty} f_n \leq a \right\} = \bigcup_{m=1}^{\infty} \left\{ x \mid \limsup_{n \to \infty} f_n < a + \frac{1}{m} \right\}
\]
are also measurable. Equip \( \mathbb{R}^* = \mathbb{R} \cup \{-\infty, \infty\} \) with the distance function
\[
d (y, z) = |g (y) - g (z)|, \quad g (t) = \frac{t}{1 + |t|}, g (-\infty) = -1, g (\infty) = 1.
\]

Open sets in \( \mathbb{R}^* \) are countable unions of open intervals, including intervals of the form \( (a, \infty] \) and \( (-\infty, a) \). Thus, \( (\limsup_{n \to \infty} f_n (x))^{-1} (U) \) is measurable for any set \( U \) open in \( \mathbb{R}^* \). By M2, the function \( \limsup_{n \to \infty} f_n : X \to \mathbb{R}^* \) is measurable. Considering \( -f_n \) instead of \( f_n \), we see that \( \liminf_{n \to \infty} f_n \) is also measurable. Since \( \mathbb{R}^* \) is separable, the map
\[
x \to \left( \liminf_{n \to \infty} f_n (x), \limsup_{n \to \infty} f_n (x) \right)
\]
is measurable from \( X \) to \( \mathbb{R}^* \times \mathbb{R}^* \) by M3. Now the set of all \( x \) such that \( \{ f_n (x) \} \) converges is the set of all \( x \) such that
\[
G \left( \liminf_{n \to \infty} f_n (x), \limsup_{n \to \infty} f_n (x) \right) = 0
\]
where \( G : \mathbb{R}^* \times \mathbb{R}^* \to \mathbb{R}^* \), \( G (A, B) = 0 \) if \( A = B \in R, 1 \) otherwise, is measurable.

Note that the argument is significantly complicated by the need to consider the case when \( \liminf_{n \to \infty} f_n (x) \) or \( \limsup_{n \to \infty} f_n (x) \) are infinite.