Find the maximum value of \(\frac{x + 2y + 3z}{\sqrt{x^2 + y^2 + z^2}}\) as \((x, y, z)\) varies among nonzero points in \(\mathbb{R}^3\).

We consider \(\mathbb{R}^3\) with the Euclidean inner product \(\langle \cdot, \cdot \rangle\) and its associated norm \(\| \cdot \|\).

Let \(u = \begin{pmatrix} x \\ y \\ z \end{pmatrix}\) and \(v = \begin{pmatrix} 1 \\ 2 \\ 3 \end{pmatrix}\).

We see that, in this setting, we have:

\[
\frac{x + 2y + 3z}{\sqrt{x^2 + y^2 + z^2}} = \frac{\langle u, v \rangle}{\| u \|}.
\]

Therefore the question asks to find

\[
\max_{u \in \mathbb{R}^3, u \neq 0} \frac{\langle u, v \rangle}{\| u \|}
\]

where \(v\) is as defined above.

On the one hand, let \(u \in \mathbb{R}^3\) nonzero. By Cauchy-Schwartz inequality, we know that

\[
\langle u, v \rangle \leq \| u \| \cdot \| v \|.
\]

So we have

\[
\frac{\langle u, v \rangle}{\| u \|} \leq \| v \|.
\]

Since the previous inequality was established for an arbitrary nonzero \(u \in \mathbb{R}^3\), we have

\[
\max_{u \in \mathbb{R}^3, u \neq 0} \frac{\langle u, v \rangle}{\| u \|} \leq \| v \|.
\]

On the other hand, consider the case where \(u = v\). We get

\[
\frac{\langle v, v \rangle}{\| v \|} = \| v \|.
\]

So

\[
\max_{u \in \mathbb{R}^3, u \neq 0} \frac{\langle u, v \rangle}{\| u \|} \geq \| v \|.
\]

We conclude that

\[
\max_{u \in \mathbb{R}^3, u \neq 0} \frac{\langle u, v \rangle}{\| u \|} = \| v \|.
\]

In our case, we note that \(\| v \| = \sqrt{14}\).

So the answer is

\[
\max_{(x, y, z) \in \mathbb{R}^3 \setminus \{0\}} \frac{x + 2y + 3z}{\sqrt{x^2 + y^2 + z^2}} = \sqrt{14}.
\]

Note: Equivalently, the question asks to find

\[
\max_{u \in \mathbb{R}^3, \| u \| = 1} \langle u, v \rangle.
\]

(The previous statement is not that obvious so think about it.) So, for a fixed \(v\), we are asked to maximize \(\langle u, v \rangle\) when \(u\) is on the unit sphere. The existence of a maximum for this problem can be concluded with the
Extreme Value Theorem (Theorem 11.22 page 301) since $\langle u, v \rangle$ is a continuous function of $u$ and the unit sphere is (sequentially) compact.

Note: In the second part, we consider the case where “$u = v$”. Why? Because we want to maximize $\langle u, v \rangle$ and we know that the inner product is maximum when the two vectors are colinear. The previous sentence is rigorously explained in Fitzpatrick EX.10.1.6.