1 Identifying types of differential equations

In this course you need to be able to identify and solve differential equations of the following types:

- separable, standard form \( f(y)y' = g(x) \)
- linear \( y' + P(x)y = Q(x) \), \( P(x), Q(x) \) are functions not containing \( y \)
- homogeneous (please see a separate document with examples on how to identify homogeneous DEs)
- Bernoulli \( y' + P(x)y = Q(x)y^n, n \neq 0, 1 \), \( P(x), Q(x) \) are functions not containing \( y \)

Each type of differential equation is identified with a standard form of the equation. Being able to convert a DE into standard form is a crucial step in its solution. The solution methods you are taught only apply to standard forms.

The first and foremost thought to keep in your head is that a differential equation may be of multiple types and thus can be converted to multiple respective standard forms.

NOTE: Bernoulli and linear differential equation types are mutually exclusive. If the DE is of one of these types, it cannot be of the other type. Think about this and satisfy yourself it is true.

The examples below should help you to get a clearer idea of the thought processes involved in identifying different DE types.

1.1 Example 1

The differential equation

\[ y' = \frac{y}{x} \]

is of three of the above types:

- separable, standard form \( \frac{y'}{y} = \frac{1}{x} \)
- linear, standard form \( y' - \frac{1}{x}y = 0 \), where \( P(x) = -\frac{1}{x} \) and \( Q(x) = 0 \)
- homogeneous, right hand side is already in the form \( F(\frac{y}{x}) \)
If you see all of the three types, you can select the quickest and easiest path to the solution: direct integration of both sides (using the fact that the equation is separable). On the other hand, if this was a more complicated equation and you found the integral difficult, you could try to use a substitution to simplify your work (homogeneous eq) or to use an integrating factor.

1.2 Example 2

If you do not convert the equation to standard form, you will not be able to apply the solution method correctly.

Let’s try to solve the linear differential equation

$$xy' + y = x.$$ 

If you forget to convert to standard form $y' + P(x)y = Q(x)$ but leave the equation as is, you may think that $P(x) = 1$ because of the term $y$.

Let’s try this and compute the integration factor: $\rho(x) = \exp(\int P(x)dx) = \exp(x) = e^x$. We proceed to multiply the entire equation by the integration factor to obtain $xe^x y' + e^x y = xe^x$. Now in the next step of the solution process, the left side is supposed to be equal to $(\rho(x)y)'$. But $(\rho(x)y)' = (e^x y)' = e^x y' + e^x y$, which is definitely not equal to the left hand side $xe^x y' + e^x y$ of the equation. Now you are stuck - the standard method of solving linear differential equations does not work. That’s because the equation was not in standard form! You can only read off $P(x)$ correctly, if you first transform the equation into standard form.

The standard form of the equation is $y' + \frac{1}{x} y = 1$. Now you see that actually, $P(x) = \frac{1}{x}$ (and $Q(x) = 1$), which makes the integration factor $\rho(x) = \exp(\int \frac{1}{x} dx) = \exp(\ln |x|) = |x|$ and we know that in case of the integration factor, we may omit the absolute values. So we multiply the DE by $x$ to obtain $xy' + y = x$, which is the derivative of the product $(\rho(x)y)$ as seen in the following $(\rho(x)y)' = (xy)' = y + xy'$, so now we can replace the left hand side to obtain the equation $(xy)' = x$ and integrate both sides with respect to $x$ to remove the derivative on the left hand side. We obtain $xy = \frac{x^2}{2} + C$ and solve for $y$ to get $y = \frac{x}{2} + Cx^{-1}$. 

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1.3 Example 3

Identify the types of the differential equation

\[ 3x^5y^2 + x^3y' = 2y^2. \]

For separability, we try to convert into the form \( f(y)y' = g(x) \), which admits direct integration or \( y' = h(y)g(x) \), whereupon we divide both sides by \( h(y) \) to separate and then integrate.

First we move everything except \( y' \) to the right side:

\[ x^3y' = 2y^2 - 3x^5y^2 \]

and

\[ y' = 2y^2x^{-3} - 3x^2y^2. \]

We see that we can factor out \( y^2 \) to obtain \( y' = y^2(2x^{-3} - 3x^2) \). The last equation is in the form \( y' = h(y)g(x) \), so we can separate

\[ \frac{y'}{y^2} = 2x^{-3} - 3x^2 \]

and integrate (this is left as an exercise to the reader, you can check your result in the book, it is problem 19 on p. 78).

The equation is not linear because when we isolate \( y' \) by dividing by \( x^3 \), we get

\[ y' + 3x^2y^2 = 2x^{-3}y^2. \]

We see that there are \( y^2 \) terms, thus the equation is not linear. However there is a good chance it might be Bernoulli precisely because of the nonlinear term \( y^2 \). We try to convert the equation to standard form for Bernoulli types: \( y' = (2x^{-3} - 3x^2)y^2 \). We see that \( P(x) = 0 \) and \( Q(x) = 2x^{-3} - 3x^2 \) and the order \( n = 2 \). Thus we can apply the standard substitution \( v = y^{1-n} = y^{-1-2} = y^{-1} \) and solve the differential equation.

The equation is not homogeneous according to our standard test. In \( y' = f(x,y) \) form, the equation reads \( y' = 2\frac{y^2}{x^3} - 3x^2y^2 \). After substitution we have \( f(tx,ty) = 2(\frac{ty^2}{(tx)^3}) - 3(tx)^2(ty)^2 = \frac{y^2}{tx^3} - 3t^4x^2y^2 \). We immediately see that the function \( f(x,y) \neq f(tx,ty) \) and thus the equation is not homogeneous.

1.4 Example 4

Identify the type of the equation

\[ y' + xy = y^2 + 4 \]

is NOT a Bernoulli equation because of the extra 4, which cannot be removed. The standard form for Bernoulli equations is \( y' + P(x)y = Q(x)y^n \), there is no place for terms other than \( y' \) or terms that are multiples of some power of \( y \).
1.5 Example 5

Is the equation $y' + xy = xy^2 + y^3$ Bernoulli? No, it isn’t because there are two different powers of $y$ present and we cannot write the equation in the standard form $y' + P(x)y = Q(x)y^n$ with just one order $n$.

1.6 Example 6

Identify the type of DE for the equation $y' + xy = y + 4$. We see only linear powers of $y$, so this could be linear. We try to convert to standard form by subtracting $y$ to obtain $y' + (x - 1)y = 4$, which is linear standard form with $P(x) = x - 1$. The equation is not separable, again because of the extra 4 term, if we try to separate by $y' = 4 - (x - 1)y$, we see we cannot factor the right hand side to obtain the form $h(y)g(x)$. The equation is not Bernoulli because the term $y'$ is isolated already and there are no powers of $y$ other than 1 present in the equation (again, since the equation is a linear DE, it cannot be a Bernoulli DE). The test using $t$ substitution (see another document on the webpage for details) can be applied and shows that the equation is not homogeneous.