Math 3000       Sample Final Exam

Name ________________________________________

Answer ten (10) of the following questions, you must choose four (4) questions from Part A, the remaining six from either part. Exam totals 150 points (15 points per question). Do your work on the blank sheets attached to these questions. Good Luck!!

Part A : You must do at least four of these.

1. If $S$ is the set of 2 element subsets of $\mathbb{N}$, prove that $S$ is denumerable.
   Every 2 element subset, $\{a,b\}$ can be thought of as an ordered pair $(a,b)$ or $(b,a)$. So the set of all 2 element subsets is a subset of $\mathbb{N} \times \mathbb{N}$ which is denumerable. An infinite subset of a denumerable set is denumerable.

2. Prove that the set of all infinite binary sequences is uncountable. [An infinite binary sequence is a never ending sequence of 0's and 1's, like $0 0 0 1 1 0 1 0 0 1 1 1 ...$ .]
   BWOC assume that this set is denumerable (it is clearly not finite). This means that the sequences can be placed in a list which will contain all such sequences. Create a new sequence by the rule, the $i$th term of the new sequence will be the opposite of the $i$th term in the $i$th sequence on the list. This new sequence can not be on the list, showing that the set of sequences is not denumerable.

3. Grade the following proof:
   Give it an "A" if the statement and proof are correct.
   Give it a "C" if the statement is correct but the proof is wrong.
   Give it an "F" if the statement and the proof are wrong.
   Justify your grade.

   **Proposition:** If $A$ and $B$ are infinite sets, then $A \sim B$. ($A$ and $B$ are equivalent).

   **Pf:** Suppose $A$ and $B$ are infinite sets. Let $A = \{a_1, a_2, a_3, \ldots\}$ and $B = \{b_1, b_2, b_3, \ldots\}$. Define $f:A \rightarrow B$ by $f(a_i) = b_i$. Then since we never run out of elements in either set, $f$ is one-to-one and onto $B$, hence $A \sim B$.

   **F.** The statement is false since, for instance, $\mathbb{N}$ and $\mathbb{R}$ are not equivalent.

4. If $S$ is any set, prove that there is no onto function $f: S \rightarrow \wp(S)$ where $\wp(S)$ is the power set of $S$.
   BWOC assume that $f: S \rightarrow \wp(S)$ is an onto function. Let $N = \{x \in S : x \notin f(x)\}$. Since $f$ is onto, there exists an $n \in S$ with $f(n) = N$. Now if $n \in N$, then by the definition of $N$, $n \notin N$. If $n \notin N$ then we have $n \in N$. This is a contradiction which shows that $f$ can not be onto.
5. Use the Schröder-Bernstein Theorem to prove the following:

For every rational number \( r \in \mathbb{Q} \), the cardinality of \( \mathbb{Q} - \{r\} \) is \( \aleph_0 \).

Let \( B = \mathbb{Q} - \{r\} \). The inclusion map \( i: B \to \mathbb{Q} \) is an injection (\( i(x) = x \)). Now define a map \( g: \mathbb{Q} \to B \) by \( g(x) = x \) if \( x < r \) and \( g(x) = x + \frac{1}{2} \) if \( x \geq r \). The image of \( g \) is contained in \( B \) and it is easily seen to be an injection. By the Schröder-Bernstein Theorem \( B \) and \( \mathbb{Q} \) are equivalent, so they have the same cardinality, \( \aleph_0 \).

6. Let \( A \subseteq B \) where \( B \) is a poset with a partial order \( R \). Prove that \( \text{sup}(A) \) is unique if it exists.

Suppose that \( m \) and \( n \) are two least upper bounds for the set \( A \). Since \( m \) and \( n \) are both upper bounds and \( m \) is a least upper bound, we have \( m R n \). Similarly, since \( n \) is a least upper bound we have \( n R m \). By the antisymmetry property of \( R \), \( n = m \).

7. Given that \( f: A \to B \) is a bijection and that \( g: B \to C \) is a bijection, prove that \( g \circ f : A \to C \) is a bijection.

Suppose that \( g \circ f (x) = g \circ f (y) \). Then, \( g(f(x)) = g(f(y)) \). But, since \( g \) is an injection, this means that \( f(x) = f(y) \). Now, since \( f \) is an injection, \( x = y \). Therefore, \( g \circ f \) is an injection. Let \( c \in C \). Since \( g \) is a surjection, there is an element \( b \in B \) with \( g(b) = c \). Since \( f \) is a surjection, there is an element \( a \in A \) such that \( f(a) = b \). But then, \( g \circ f (a) = g(f(a)) = g(b) = c \), so, since \( c \) was arbitrary, \( g \circ f \) is a surjection. Therefore, \( g \circ f \) is a bijection.

8. If \( \mathbb{N} \to \mathbb{Z} \) is given by \( f(x) = x^2 \), show that \( f \) is one-one but not onto \( \mathbb{Z} \). (\( \mathbb{N} \) is the set of natural numbers and \( \mathbb{Z} \) is the set of integers.)

If \( f(x) = f(y) \), then \( x^2 = y^2 \) which implies that \( |x| = |y| \). Since \( x \) and \( y \) are natural numbers and thus positive, this means that \( x = y \) and \( f \) is one-to-one. The range of \( f \) consists of positive squares, so an integer such as \( 3 \) would not be in the range and \( f \) is not onto.

9. Let \( A \) be a set. Consider the partial order \( \subseteq \) on \( \mathcal{P}(A) \), the power set of \( A \). If \( C \) and \( D \) are subsets of \( A \), prove that the greatest lower bound of \( \{C,D\} \) is \( C \cap D \).

Since \( C \cap D \subseteq C \) and \( C \cap D \subseteq D \), we have that \( C \cap D \) is a lower bound of \( \{C,D\} \). If \( T \) is any lower bound of \( \{C,D\} \), then \( T \subseteq C \) and \( T \subseteq D \) so \( T \subseteq C \cap D \) and so \( C \cap D \) is a greatest lower bound of \( \{C,D\} \).

10. Define the following terms (do not just indicate notation):

a. Given a set \( S \) with a partial order \( R \), and a subset \( B \) of \( S \), a **lower bound** of \( B \) is ... an element \( t \) of \( S \) for which \( t R b \) for all \( b \) in \( B \).

b. A **partition** of a set \( S \) is ... a family of non-empty subsets of \( S \) which are mutually disjoint and whose union is \( S \).

c. A **function** \( f: A \to B \) is a relation between \( A \) and \( B \) such that ... each element of \( A \) appears exactly once as a first coordinate in the ordered pairs of the relation.

d. A set \( S \) is **denumerable** if ... it is equivalent to the set of natural numbers.

e. A **partial order** on a set \( S \) is a ... relation on \( S \) which is reflexive, antisymmetric and transitive.

f. A function \( f: A \to B \) is a **surjection** (onto) if ... the range of \( f \) equals the codomain (\( B \)).

g. The cardinality of the set of natural numbers \( \mathbb{N} \) is denoted by the symbol .... \( \aleph_0 \).

h. A relation \( R \) on a set \( S \) is **reflexive** if ... \( x R x \) for all \( x \) in \( S \).

i. Two sets \( A \) and \( B \) are said to be **equivalent** if ... there is a bijection between them.
j. The supremum of a subset C of a partially ordered set S, with partial order R, is an upper bound t of C for which t R u for every upper bound u of C.

k. A subset of real numbers is **closed** if its complement is an open set.

**Part B:**

11. If \( P(x) \) and \( Q(x) \) are propositional functions involving \( x \), show that
\[
(\forall x)(P(x) \Rightarrow Q(x)) \Rightarrow (\exists x)P(x) \Rightarrow (\exists x)Q(x).
\]
Since \( (\exists x)P(x) \) we may assume that \( P(z) \) is true. Since \( (\forall x)(P(x) \Rightarrow Q(x)) \), we also have \( Q(z) \). Therefore, \( (\exists x)Q(x) \) is true. So, \( (\exists x)P(x) \Rightarrow (\exists x)Q(x) \).

12. Let A and B be subsets of a universal set X. Prove that
\[
(A - B) \cup (B - A) = (A \cup B) - (A \cap B).
\]
Proof: \( (A - B) \cup (B - A) = (A \cap B) \cup (B \cap A) = ((A \cap B) \cup B) \cap ((A \cap B) \cup \overline{A}) \)
\[
= ((A \cup B) \cap \overline{B}) \cap ((A \cup \overline{A}) \cap (B \cup \overline{A}))
\]
\[
= ((A \cup B) \cap \overline{B}) \cap (X \cap (B \cup \overline{A}))
\]
\[
= (A \cup B) \cap (\overline{B} \cap \overline{B}) \cap (X \cap (B \cup \overline{A}))
= (A \cup B) \cap (B \cap A)
= (A \cup B) - (A \cap B).
\]

13. A cafeteria has 3 meat selections, 4 vegetable selections and 5 dessert selections. Johnny, who has a sweet tooth, will choose a meal consisting of 1 meat selection, 2 vegetable selections (his mother made him do it) and 2 dessert selections. How many different meals can Johnny create? Keep in mind that Johnny may like a dessert so much that he may choose two portions of it, but he would never select two portions of any vegetable. [Show work, the answer alone will not get much credit]
\[
C(3,1)C(4,2)[C(5,2) + C(5,1)] = (3)(6)(10+5) = 270.
\]

14. Prove that for any sets \( A, B \) and \( C \),
\[
(A \cap B) \times C = (A \times C) \cap (B \times C).
\]
If \( (x,y) \in (A \cap B) \times C \), then \( x \in A \) and \( x \in B \) and \( y \in C \). Thus, \( (x,y) \in A \times C \) and \( (x,y) \in B \times C \), so \( (A \cap B) \times C \subseteq (A \times C) \cap (B \times C) \). On the other hand, if \( (x,y) \in (A \times C) \cap (B \times C) \) then \( x \in A \) and \( x \in B \) and \( y \in C \), so \( (x,y) \in (A \cap B) \times C \) and we have \( (A \cap B) \times C \subseteq (A \times C) \cap (B \times C) \), so we have equality.

15. Prove by induction that
\[
n^3 + 5n + 6 \text{ is divisible by 3 for all natural numbers } n.
\]
Let \( S = \{ n : 3 \text{ divides } n^3 + 5n + 6 \} \). Since \( (1)^3 + 5(1) + 6 = 12 \) which is divisible by 3, we have \( 1 \in S \). Consider \( (n+1)^3 + 5(n+1) + 6 = n^3 + 5n + 6 + 3n^2 + 3n + 1 + 5 = n^3 + 5n + 6 + 3(n^2 + n + 2) \) and so is divisible by 3. Thus, \( n + 1 \in S \) and by the PMI the result follows.

16. Define a relation \( R \) on the set \( N \) of natural numbers by
\[
(n,m) \in R \text{ means that } n^2 - m^2 \text{ is a multiple of 5, show that } R \text{ is an equivalence relation and describe the equivalence class which contains the number } 2.
\]
\((n,n) \in R \text{ for all } n \text{ since } n^2 - n^2 = 0, \text{ a multiple of 5. So } R \text{ is reflexive. If } (n,m) \in R, \text{ then } n^2 - m^2 \text{ is a multiple of 5, so } m^2 - n^2 = -(n^2 - m^2) \text{ is also a multiple of 5. Thus, } (m,n) \in R \text{ and } R \text{ is symmetric. If } (n,m) \in R \text{ and } (m,s) \in R \text{ then } n^2 - m^2 = 5k \text{ and } m^2 - s^2 = 5u, \text{ so } n^2 - s^2 = 5(k + u) \text{ and thus } (n,s) \in R \text{ and } R \text{ is transitive. } R \text{ is an equivalence relation.}
17. If S is a finite set with n elements prove that the power set of S has $2^n$ elements.

Let $T = \{ n \in \mathbb{N} | \text{the statement is true} \}$

If $|S| = 1$, then $S = \{a\}$ for some $a$ and $\mathcal{P}(S) = \{\emptyset, S\}$ which has $2^1 = 2$ elements. Now assume that $k \in T$. Consider a set $S$ with $|S| = k+1$. Let $x$ be a specific member of $S$. The subsets of $S$ either contain $x$ or don't. Let $S' = S - \{x\}$. Then $|S'| = k$. A subset of $S$ which does not contain $x$ is a subset of $S'$, so there are $2^k$ of them since $k \in T$. Those that do contain $x$ consist of $x$ together with a subset of $S'$, so again there are $2^k$ of them. Therefore, there are $2^k + 2^k = 2^{k+1}$ subsets of $S$ and so $k+1 \in T$. By the PMI, $T$ is the set of all natural numbers and the theorem is proved.

18. Consider the set $\mathbb{N}_m = \{1, 2, \ldots, m\}$. If $\frac{1}{4}$ of all the 3 element subsets of $\mathbb{N}_m$ contain the element 7, what is $m$?

The number of 3 element subsets is $C(m,3)$ if a 3 element subset of $\mathbb{N}_m$ contains 7, then it contains 2 other elements chosen from $m - 1$ possibilities (can't pick a 7). Thus, we have the equation,

$$\frac{1}{4} C(m,3) = C(m-1, 2).$$

So,

$$\frac{1}{4}(1/6)(m)(m-1)(m-2) = (1/2)(m-1)(m-2)$$

from which we solve for $m$, to get $m = 12$.

19. Prove that for any sets $A$ and $B$,

$$(A - B)^c = A^c \cup B$$

where $X^c$ is the complement of the set $X$.

Proof: $(A - B) = (A \cap \overline{B}) = \overline{A \cup B} = \overline{A} \cup B$.

20. The completeness property of the reals states that if a set $A$ of real numbers has an upper bound, then it has a least upper bound (sup($A$)). Use this to prove that if a set $B$ of real numbers has a lower bound, then it has a greatest lower bound (inf($B$)).

Let $X$ be the set of all lower bounds of $B$. $X$ is not empty and any element of $B$ is an upper bound of $X$, so by the completeness property $X$ has a least upper bound, sup($X$). Now sup($X$) is a lower bound of $B$, for if not there would exist an element $b$ in $B$ with $b <$ sup($X$), but $b$ is an upper bound of $X$ violating sup($X$) being the least upper bound of $X$. If $t$ is any lower bound of $B$, then $t$ is in $X$ and $t \leq$ sup($X$), so sup($X$) is the greatest lower bound of $B$. 

\[\{n : (n,2) \in \mathbb{R} \} = \{n : n^2 - 4 = 5k \text{ for some integer } k\} = \{2, 3, 7, 8, 12, 13, \ldots\}.\]